CONSTRUCTING VECTOR-VALUED AUTOMORPHIC FORMS ON UNITARY GROUPS

THOMAS L. BROWNING, PAVEL ČOUPEK, ELLEN EISCHEN, CLAIRE FRECHETTE,
SERIN HONG, SI YING LEE, AND DAVID MARCIL

ABSTRACT. We introduce a method for producing vector-valued automorphic forms on unitary groups from scalar-valued ones. As an application, we construct an explicit example. Our strategy employs certain differential operators. It is inspired by work of Cléry and van der Geer in the setting of Siegel modular forms, but it also requires overcoming challenges that do not arise in the Siegel setting.

1. INTRODUCTION

Automorphic forms play a key role in number theory. Automorphic forms on unitary groups have proved to be particularly valuable, thanks to structures that arise in this setting. Producing explicit examples of automorphic forms on unitary groups remains challenging, though, and there are relatively few such examples in the literature.

In the setting of unitary groups, one must work with not only scalar-valued but also vector-valued automorphic forms. We introduce a method for constructing vector-valued automorphic forms on unitary groups from scalar-valued ones. As an application, we construct an explicit example. Our strategy employs certain differential operators.

Our approach is inspired by work Cléry and van der Geer carried out for Siegel modular forms, i.e. automorphic forms on symplectic groups [CvdG15]. Their work extends a strategy of Witt [Wit41]. Unitary groups bear certain similarities to symplectic groups. The setting of unitary groups also presents new challenges, though, which we overcome in this paper. Related to this, the literature has many more explicit examples for Siegel modular forms than for automorphic forms on unitary groups. This paper achieves three goals:

(1) Extend Cléry and van der Geer's strategy using differential operators [CvdG15] to unitary groups of all signatures (Proposition 2.12 and Theorem 2.14).
(2) Apply this approach in an explicit example (Theorem 3.7), which does not carry over trivially from the Siegel case and illustrates challenges new to this setting.
(3) Provide a coordinate-free, geometric formulation of our construction (Theorem 5.10). While unnecessary for our other goals, this is a more intrinsic approach.

1.1. **Summary of main results and relationship with earlier work.** We draw inspiration from the aforementioned methods that [CvdG15, Wit41] introduced for Siegel modular forms for $Sp(2g, \mathbb{Z})$. Those forms are defined on Siegel upper half-space

$$\mathfrak{H}_g = \{\tau \in \mathrm{Mat}_{g \times g}(\mathbb{C}) \mid {}^t\tau = \tau \text{ with } \mathrm{Im}(\tau) \text{ positive-definite}\}.$$

Consider the restriction of a Siegel modular form f for $Sp(2g, \mathbb{Z})$ via the embedding

$$\mathfrak{H}_j \times \mathfrak{H}_{g-j} \to \mathfrak{H}_g \qquad \text{given by} \qquad (\tau', \tau'') \mapsto \begin{pmatrix} \tau' & 0 \\ 0 & \tau'' \end{pmatrix}$$

for some $0 \le j < g$. We write an arbitrary element $\tau \in \mathfrak{H}_g$ as

$$\tau = \begin{pmatrix} \tau' & x \\ {}^t x & \tau'' \end{pmatrix}.$$

If f vanishes to order r on $\mathfrak{H}_j \times \mathfrak{H}_{g-j}$, then a certain restricted differential form $\mathrm{d}_x^r f|_{\mathfrak{H}_j \times \mathfrak{H}_{g-j}}$ decomposes into tensor products of Siegel modular forms on $\mathfrak{H}_j$ and $\mathfrak{H}_{g-j}$ [CvdG15, Propositions 2.2 and 2.3] and can be used to produce explicit vector-valued Siegel modular forms from scalar-valued ones [CvdG15, Section 3].

How, if at all, does this extend to the setting of automorphic forms on a unitary group G? When G is of signature (n, n), similarities with the case of symplectic groups suggest a starting point. Working with other signatures is more complicated. Our results in this paper are completely general, in the sense that we handle all signatures. Given unitary groups U_α and U_β with an embedding $U_\alpha \times U_\beta \hookrightarrow U$ for U a larger unitary group, we have a corresponding embedding of symmetric spaces $\mathcal{H}_\alpha \times \mathcal{H}_\beta \hookrightarrow \mathcal{H}$ analogous to the embedding of Siegel upper half-spaces above. In this case, $\tau \in \mathcal{H}$ is given by

$$\tau = \begin{pmatrix} \tau_\alpha & x \\ y & \tau_\beta \end{pmatrix}$$

with $\tau_\alpha \in \mathcal{H}_\alpha$ and $\tau_\beta \in \mathcal{H}_\beta$. Our first main result, Theorem 2.14, extends [CvdG15, Propositions 2.2 and 2.3] to unitary groups of all signatures and is summarized here:

Theorem A (Summary of Theorem 2.14)**.** Suppose f is a scalar-valued automorphic form on the unitary group U that vanishes to order r on $\mathcal{H}_\alpha \times \mathcal{H}_\beta$.

(1) Restricted differential forms $\mathrm{d}_x^r f|_{\mathcal{H}_\alpha \times \mathcal{H}_\beta}$ and $\mathrm{d}_y^r f|_{\mathcal{H}_\alpha \times \mathcal{H}_\beta}$ decompose into sums of tensor products of automorphic forms on the unitary groups U_α and U_β.

(2) If f is a cusp form, so are all automorphic forms appearing in (1).

It is straightforward to recover the approach in [CvdG15, Wit41] as a special case of the more general construction in this paper. The geometric reformulation of our operators (Section 5, which culminates with Theorem 5.10) also suggests that this general construction could be extended still further. This would likely come at the cost, though, of not getting the sort of explicit example we now describe.

As an application of Theorem 2.14, we produce an explicit example of a vector-valued automorphic form for unitary groups. Inspiration comes from [CvdG15, Section 3], which produces vector-valued Siegel modular forms from derivatives of the *Schottky form*. The Schottky form is a Siegel modular form J on $\mathfrak{H}_4$, defined either as the Ikeda lift of the discriminant modular form Δ, or as a difference between the theta

series attached to the unimodular lattices $E_8 \oplus E_8$ and D_{16}^+. Cléry and van der Geer explicitly describe the Siegel modular forms $\mathrm{d}_x^4 J|_{\mathfrak{H}_2 \times \mathfrak{H}_2}$ and $\mathrm{d}_x^4 J|_{\mathfrak{H}_3 \times \mathfrak{H}_1}$:

(1) J vanishes to order 4 on $\mathfrak{H}_2 \times \mathfrak{H}_2$ with $\mathrm{d}_x^4 J|_{\mathfrak{H}_2 \times \mathfrak{H}_2} = \chi_1 \otimes \chi_1$ for some (scalar-valued) cusp form χ_1 on $\mathfrak{H}_2$.
(2) J vanishes to order 4 on $\mathfrak{H}_3 \times \mathfrak{H}_1$ with $\mathrm{d}_x^4 J|_{\mathfrak{H}_3 \times \mathfrak{H}_1} = \chi_2 \otimes \Delta$ for some (vector-valued) cusp form χ_2 on $\mathfrak{H}_3$.

The scalar form χ_1 generates the space of cusp forms on $\mathfrak{H}_2$ of weight 10. The vector-valued form χ_2 generates the space of cusp forms on $\mathfrak{H}_3$ for a specified vector weight.

In this paper, we consider an analogue of the Schottky form, namely the *Hermitian Schottky form* that Hentschel and Krieg defined on a unitary group of signature $(4, 4)$ [HK06]. For the moment, to highlight the relationship with J, we denote this form by $\tilde{J}$. The restriction of this form to $\mathcal{H}_4$ is J. It is tempting to assume all the results for the Siegel case will carry over to this setting, but that does not quite turn out to be the case. We obtain Theorem 3.7, which concerns forms on Hermitian symmetric spaces $\mathcal{H}_n$ for unitary groups of signature (n, n) and relies on an embedding $\mathcal{H}_j \times \mathcal{H}_{n-j} \hookrightarrow \mathcal{H}_n$ analogous to the embedding of Siegel upper-half spaces above.

Theorem B (Summary of Theorem 3.7). The scalar form $\tilde{J}$ vanishes to order 4 on $\mathcal{H}_3 \times \mathcal{H}_1$. Furthermore, the form $\mathrm{d}_x^4 \tilde{J}|_{\mathcal{H}_3 \times \mathcal{H}_1}$, whose weight is specified in Expression (10), can be written as $M \otimes \Delta$ for some (vector-valued) cusp form M on $\mathcal{H}_3$.

Remark 1.1. The Fourier coefficients of $M \otimes \Delta$ can be explicitly computed, as seen in the proof of Proposition 3.6. However, this is only practical for the first few coefficients.

Key differences between Theorem 3.7 and the corresponding construction for the Schottky form J include:

- The form $\tilde{J}$ does not vanish at all on $\mathcal{H}_2 \times \mathcal{H}_2$.
- We do not claim that M generates the entire space of (vector-valued) cusp forms on $\mathcal{H}_3$ of its weight.

These differences reflect some new challenges that arise in the unitary setting. The first point indicates that the order of vanishing for a modular form does not behave well under the natural embedding of the Siegel space into the Hermitian space. The second point is related to the fact that the dimension of the space parametrizing cusp forms of specified weight currently remains unknown in the unitary setting, in contrast to the Siegel setting. Consequently, in contrast to the frequent reliance on dimension formulas for cusp forms in the Siegel setting in [CvdG15], our proof of Theorem 3.7 does not use any dimension formulas. Instead, we rely on the computation of Fourier coefficients for the derivatives of $\tilde{J}$.

Remark 1.2. While we focus on a particular construction in this paper, it would be interesting to explore the relationship between the differential operators here and others that have been constructed for symplectic and unitary groups, such as Maass–Shimura operators [Har79, Har81a, Har81b, Shi81] and Rankin–Cohen brackets [Ban06, Dun24, MS17, EI98, Ibu99].

1.2. **Organization of the paper.** Section 2 introduces automorphic forms on unitary groups and certain differential operators. This section then presents our main results about differential operators, Proposition 2.12 and Theorem 2.14, producing vector-weight forms from scalar-weight ones. In Section 3, we apply the operators in an explicit example. The main result of this section is Theorem 3.7. Section 4 presents proofs of the results from Section 2. Finally, Section 5 provides a geometric reformulation of the operators. While this geometric portion is unnecessary for the results earlier in the paper, it is likely to be of interest to those working with Shimura varieties or seeking an intrinsic, geometric understanding of the operators from Section 2.

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2. Automorphic forms and differential operators

We begin by introducing automorphic forms on unitary groups, and we construct certain differential operators that act on them. In this section, we state our main results in a direct (coordinate-dependent) way, because this will be best suited to our application concerning an explicit example in the following section. The proofs of the main assertions (Proposition 2.12 and Theorem 2.14) will be postponed to Section 4, after an example in Section 3. For a more comprehensive treatment of automorphic forms on unitary groups from the perspectives employed in this paper, the reader might also consult [Eis24].

2.1. **Complex automorphic forms.** Firstly, we specify notation and conventions for holomorphic automorphic forms on unitary groups considered.

Let $K/\mathbb{Q}$ be an imaginary quadratic number field, and let c be the unique nontrivial element of $\mathrm{Gal}(K/\mathbb{Q})$. Given $k \in K$, we set $\overline{k} := c(k)$. Consider a finite-dimensional K-vector space V equipped with a non-degenerate Hermitian pairing $\langle \cdot, \cdot \rangle : V \times V \to K$.

Definition 2.1. The *unitary group* $\mathrm{U}(V) = \mathrm{U}(V, \langle \cdot, \cdot \rangle)$ is the algebraic group over $\mathbb{Q}$ whose points are given by

$$\mathrm{U}(V)(A) = \{g \in \mathrm{GL}(V \otimes_{\mathbb{Q}} A) \mid \langle gv, gw \rangle = \langle v, w \rangle \text{ for all } v, w \in V \otimes_{\mathbb{Q}} A\}, \quad A \in \mathsf{Alg}_{\mathbb{Q}}.$$

More generally, the *similitude unitary group* $\mathrm{GU}(V) = \mathrm{GU}(V, \langle \cdot, \cdot \rangle)$ is given by

$$\mathrm{GU}(V)(A) = \{(g, \nu(g)) \in \mathrm{GL}(V \otimes_{\mathbb{Q}} A) \times A^{\times} \mid \langle gv, gw \rangle = \nu(g)\langle v, w \rangle\}, \quad A \in \mathsf{Alg}_{\mathbb{Q}}.$$

The map $\nu : \mathrm{GU}(V) \to \mathbb{G}_{\mathrm{m}}$ given by $(g, \nu(g)) \mapsto \nu(g)$ is a group homomorphism, with $\mathrm{Ker}\, \nu = \mathrm{U}(V)$. While $\mathrm{GU}(V)$ will play a role later in Section 5 for algebro-geometric interpretation, for our present purposes it suffices to work with the group $\mathrm{U}(V)$ only.

Denote by $\mathbb{A}_f$ the ring of finite adèles of $\mathbb{Q}$. A *congruence subgroup* $\Gamma \subseteq \mathrm{U}(V)(\mathbb{Q})$ is a subgroup of the form

$$\Gamma = \mathrm{U}(V)(\mathbb{Q}) \cap \mathcal{U}$$

for some open compact subgroup $\mathcal{U} \subseteq \mathrm{U}(V)(\mathbb{A}_f)$. Equivalently, upon choosing an integral model $\mathcal{U}(V)$ of $\mathrm{U}(V)$, Γ is a subgroup of $\mathrm{U}(V)(\mathbb{Q})$ that contains the principal congruence subgroup

$$\Gamma(N) = \{g \in \mathcal{U}(V)(\mathbb{Z}) \mid g \mapsto \mathrm{Id} \in \mathcal{U}(V)(\mathbb{Z}/N\mathbb{Z})\}$$

for some N as a finite-index subgroup (this property does not depend on a choice of integral model, see e.g. [Mil04, Section 4] for a detailed discussion).

Choosing a suitable basis of $V \otimes_{\mathbb{Q}} \mathbb{R}$, the pairing $\langle \cdot, \cdot \rangle$ can be represented by the matrix

$$I_{m,n} = \begin{bmatrix} \mathrm{Id}_m & 0 \\ 0 & -\mathrm{Id}_n \end{bmatrix}$$

for some pair of integers (m, n). In this case, $\mathrm{U}(V)(\mathbb{R})$ can be identified with the Lie group

$$\mathrm{U}(m, n) = \{\gamma \in \mathrm{GL}_d(\mathbb{C}) \mid {}^t\overline{\gamma} I_{m,n} \gamma = I_{m,n}\}.$$

We call the pair (m, n) the *signature* of $\mathrm{U}(V)$.

The group $\mathrm{U}(m, n)$ naturally acts on the *bounded Hermitian space*

$$\mathcal{H}_{m,n} = \{\tau \in \mathrm{Mat}_{m \times n}(\mathbb{C}) \mid \mathrm{Id}_n - {}^t\overline{\tau}\tau \text{ is positive-definite}\}$$

via linear fractional transformations, i.e.

$$\gamma\tau = (A\tau + B)(C\tau + D)^{-1}, \quad \tau \in \mathcal{H}_{m,n}, \quad \gamma = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathrm{U}(m, n)$$

(where the sizes of the blocks are determined by A being $m \times m$ and D being $n \times n$).

Given $\gamma \in \mathrm{U}(V)(\mathbb{R})$, identified with $\mathrm{U}(m, n)$ as above, and $\tau \in \mathcal{H}_{m,n}$, we define the *automorphy factors* $\lambda_\gamma(\tau) \in \mathrm{GL}_m(\mathbb{C})$, $\mu_\gamma(\tau) \in \mathrm{GL}_n(\mathbb{C})$ as follows:

$$\lambda_\gamma(\tau) = \overline{B}({}^t\tau) + \overline{A}, \quad \mu_\gamma(\tau) = C\tau + D, \quad \gamma = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathrm{U}(m, n).$$

With this setup, we employ the following definition of automorphic forms:

Definition 2.2. Let $\mathrm{U} = \mathrm{U}(V)$ be a unitary group of signature (m, n), let $\Gamma \subseteq \mathrm{U}(\mathbb{Q})$ be a congruence subgroup, and consider a representation (ρ, W) of $\mathrm{GL}_m(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C})$. An *automorphic form of level Γ and weight ρ* is a holomorphic map $f : \mathcal{H}_{m,n} \to W$ satisfying

$$(1) \qquad f(\tau) = (f||_\rho \gamma)(\tau) := \rho\left(\lambda_\gamma(\tau), \mu_\gamma(\tau)\right)^{-1} f(\gamma\tau), \quad \tau \in \mathcal{H}_{m,n}, \quad \gamma \in \Gamma.$$

When the signature is $(1, 1)$ and U is quasi-split over $\mathbb{Q}$, we additionally require that f is holomorphic at all cusps. Denote by $M_\rho(\Gamma)$ the space of all automorphic forms of weight ρ and level Γ.

Remark 2.3. We defer the discussion of holomorphicity at cusps to Remark 2.6, after introducing a variant of automorphic forms (amounting to a coordinate change) that is more suitable for the description of the conditon. For now let us only remark that in

all the cases except of quasi–split unitary group over $\mathbb{Q}$ of signture $(1,1)$, the analogue of the condition is automatically satisfied by Koceher's principle [Lan16].

Example 2.4. In the case of the representation

$$\Delta_{k,l} = \det^k \boxtimes \det^l : \mathrm{GL}_m(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C}) \to \mathbb{C} \otimes \mathbb{C} \simeq \mathbb{C},$$

we denote the space of automorphic forms of weight ρ (and level Γ) also by $M_{(k,l)}(\Gamma)$, and refer to its elements as automorphic forms *of weight* (k, l).

Furthermore, we refer to such automorphic forms as *scalar-valued*, and to general automorphic forms as *vector-valued* when emphasising the distinction.

We sometimes write $\Delta_{k,l}$ as $\Delta_{(m,n),(k,l)}$ when we wish to emphasize the choice of ranks for both general linear groups involved in the definition.

2.2. **Variant: Hermitian modular forms.** Motivated by the example discussed in Section 3, we consider the following variant. Suppose $\mathrm{U} = \mathrm{U}(V)$ is of equal signature, i.e. $m = n$, and let us additionally assume that U is quasi-split over $\mathbb{Q}$. Then a suitable choice of basis of $V \otimes_{\mathbb{Q}} \mathbb{R}$ (in fact, of V by the quasi-splitness assumption) allows one to express the pairing $\langle \cdot, \cdot \rangle$ by the matrix $i\eta_n$ where

$$\eta_n = \begin{bmatrix} & -\mathrm{Id}_n \\ \mathrm{Id}_n & \end{bmatrix},$$

identifying $\mathrm{U}(\mathbb{R})$ with

$$\mathrm{U}(\eta_n) = \{\gamma \in \mathrm{GL}_{2n}(\mathbb{C}) \mid {}^t\overline{\gamma}\eta_n\gamma = \eta_n\}.$$

The group $\mathrm{U}(\eta_n)$ then naturally acts on the *unbounded Hermitian space*

$$\mathcal{H}_n = \{\tau \in \mathrm{Mat}_{n \times n}(\mathbb{C}) \mid i({}^t\overline{\tau} - \tau) \text{ is positive-definite}\}$$

again via fractional linear transformations, i.e.

$$\gamma\tau = (A\tau + B)(C\tau + D)^{-1}, \quad \tau \in \mathcal{H}_n, \quad \gamma = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathrm{U}(\eta_n).$$

We then define, for $\gamma \in \mathrm{U}(\mathbb{R})$ viewed as an element of $\mathrm{U}(\eta_n)$, the automorphy factors $\lambda_\gamma(\tau), \mu_\gamma(\tau) \in \mathrm{GL}_n(\mathbb{C})$ as follows:

$$\lambda_\gamma(\tau) = \overline{C}({}^t\tau) + \overline{D}, \quad \mu_\gamma(\tau) = C\tau + D, \quad \gamma = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathrm{U}(\eta_n).$$

Definition 2.5. Let $\mathrm{U} = \mathrm{U}(V)$ be a unitary group of signature (n, n) and quasi-split over $\mathbb{Q}$. Consider a congruence subgroup $\Gamma \subseteq \mathrm{U}(\mathbb{Q})$ and a representation (W, ρ) of $\mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C})$. A *Hermitian modular form of level* Γ *and weight* ρ is a holomorphic map $f : \mathcal{H}_n \to W$ satisfying

$$f(\tau) = (f||_\rho\gamma)(\tau) := \rho\left(\lambda_\gamma(\tau), \mu_\gamma(\tau)\right)^{-1} f(\gamma\tau), \quad \tau \in \mathcal{H}_n, \quad \gamma \in \Gamma.$$

Moreover, when the signature is $(1, 1)$, we additionally require that f is holomorphic at every cusp.

Remark 2.6. Let us spell out the meaning of the holomorphicity condition along the lines of [Shi00, § 5]. Given a hermitian modular function f of level Γ and weight ρ (i.e. a function satisfying the modularity condition of Definition 2.5, but not necessarily the holomorphicity at cusps), f admits a Fourier expansion of the form

$$(2) \qquad f(\tau) = \sum_{h \in M^\vee} \mathbf{c}(h) \exp(2\pi i \operatorname{tr}(h\tau))$$

where

- M is a $\mathbb{Z}$-lattice of complex Hermitian $n \times n$ matrices α with $\begin{bmatrix} \operatorname{Id}_n & \alpha \\ 0 & \operatorname{Id}_n \end{bmatrix} \in \Gamma$, i.e. such that $f(\tau + \alpha) = f(\tau)$ (existence of such lattice is guaranteed by the quasi-splitness assumption),
- $M^\vee$ is the $\mathbb{Z}$-lattice of all complex Hermitian $n \times n$ matrices h with $\operatorname{tr}(h\alpha) \in \mathbb{Z}$ for all $\alpha \in M$,
- $\mathbf{c}(h)$ are vectors in the underlying vector space W_ρ of ρ.

When $n = 1$, the Hermitian matrices $h \in M^\vee$ and the coefficients $c(h)$ are just real and complex numbers, respectively. In this case, we say that f *is holomorphic at ∞* if $c(h) = 0$ whenever $h < 0$. We say that f *is holomorphic at all cusps* if for all $\beta \in \operatorname{SL}_2(\mathbb{Q})$, the function $f||_\rho \beta$ (automorphic of level $\beta^{-1}\Gamma\beta$) is holomorphic at ∞.

We emphasize that for higher n, the analogous condition is automatic by Koecher's principle, which in the presence of Fourier coefficients can be stated as follows.

Proposition 2.7 (Koecher's principle; [Shi00, Proposition 5.7]). *When $n > 1$ and f is a hermitian modular function of some weight ρ and level $\Gamma \subseteq \operatorname{U}(\eta_n)$, in the expression (2) one has $\mathbf{c}(h) \neq 0$ only if h is positive–semidefinite.*

Fourier expansions also allow us to conveniently define hermitian cusp forms.

Definition 2.8. Given a hermitian modular form f of level Γ and weight ρ, f is called a *cusp form* if for any $\beta \in \operatorname{U}(\mathbb{Q})$, in the Fourier expansion (2) of $f||_\rho \beta$, one has $\mathbf{c}(h) = 0$ whenever h is not positive-definite.

We denote the space of all Hermitian modular forms of weight ρ and level Γ again by $M_\rho(\Gamma)$ (we hope that there is little potential for substantial confusion). Similarly, we employ the notation $M_{(k,l)}(\Gamma)$ when $\rho = \det^k \boxtimes \det^l$, and refer to Hermitian modular forms of this type as *scalar-valued*. We denote the space of all all Hermitian cusp forms by $S_\rho(\Gamma)$ (and $S_{(k,l)}(\Gamma)$ if $\rho = \det^k \boxtimes \det^l$).

Following Shimura's notation from [Shi00], when convenient to make the disctinction, we will refer to unitary groups and automorphic forms in the coordinates described in Section 2.1 as the "case (UB)" (where "UB" stands for "unitary ball") and to hermitian forms on $U(\eta_n)$ in the sense of this section as the "case (UT)" (i.e., "unitary tube").

2.3. **Further variants.** Following [Shi78], let us mention two further variants that will serve an auxiliary purpose thanks to their convenience in expressing automorphc forms via Fourier expansions. Let us fix a unitary group U of signature (m, n).

Firstly, we consider the case $m \neq n$. Without loss of generality, let us assume that $m > n$. Then $U(\mathbb{R})$ may also be realized as the group

$$\widetilde{U}(m,n) = \{g \in \mathrm{GL}_{n+m}(\mathbb{C}) \mid {}^t\overline{g}\eta_{m,n}g = \eta_{m,n}\},$$

where

$$\eta_{m,n} = \begin{bmatrix} & & \mathrm{Id}_n \\ & S & \\ -\mathrm{Id}_n & & \end{bmatrix}$$

with S diagonal skew-Hermitian and such that $-iS$ is positive-definite. $U(\mathbb{R})$ then naturally acts on the symmetric space

$$\widetilde{\mathcal{H}}_{m,n} = \left\{ \begin{bmatrix} \tau \\ u \end{bmatrix} \mid \tau \in \mathrm{Mat}_{n \times n}(\mathbb{C}), u \in \mathrm{Mat}_{(m-n) \times n}(\mathbb{C}), \quad -i(\tau - {}^t\overline{\tau}) - {}^t\overline{u}Su > 0 \right\}.$$

This is the convention taken in [Shi78] (up to order of coordinates), where the appropriate automorphy factors $\lambda_\gamma(\tau), \mu_\gamma(\tau)$ are given (in the notation of *loc. cit.*, $\kappa(\gamma, \tau)$ and $\mu(\gamma, \tau)$, resp.), and the corresponding automorphic forms are defined. As explained in *loc. cit.*, the automorphic forms admit *Fourier–Jacobi expansions*, that is, they can be written in the form

$$f(\tau, u) = \sum_h \mathbf{c}(u; h) \exp(2\pi i \, \mathrm{tr}(h\tau)),$$

with h ranging over a suitable lattice of $n \times n$ hermitian matrices and the coefficient functions $\mathbf{c}(u; h)$ are certain theta functions. Similarly to the previous case, the Koecher's principle implies that $\mathbf{c}(u; h) = 0$ unless h is non-negative (cf. [Shi78, p. 570]).

Lastly, let us consider the case $m = n > 1$, but when U is not quasi-split over $\mathbb{Q}$. By [Shi78, §6], the following form of the group can nonetheless be achieved:

$$U(\mathbb{R}) \simeq U(\widetilde{\eta_n}) = \{g \in \mathrm{GL}_{2n}(\mathbb{C}) \mid {}^t\overline{g}\widetilde{\eta_n}g = \widetilde{\eta_n}\},$$

with

$$\widetilde{\eta_n} = \begin{bmatrix} t & & \\ & & \mathrm{Id}_{n-1} \\ & s & \\ -\mathrm{Id}_{n-1} & & \end{bmatrix},$$

where $s, t \in K$ are pure imaginary elements whose product is positive. The associated symmetric space is given as

$$\widetilde{\mathcal{H}}_n = \{Z \in \mathrm{Mat}_{n \times n}(\mathbb{C}) \mid i \begin{bmatrix} {}^t\overline{Z} & \mathrm{Id}_n \end{bmatrix} \widetilde{\eta}_n \begin{bmatrix} Z \\ \mathrm{Id}_n \end{bmatrix} > 0\}.$$

Automorphic forms on $\widetilde{H}_n$ admit Fourier–Jacobi expansions of the form

$$f(\tau) = \sum_h \mathbf{c}(u, v, w; h) \exp(2\pi i \, \mathrm{tr}(h\tau')),$$

where we consider the coordinates

$$\widetilde{H}_n \ni \tau = \begin{bmatrix} u & v \\ w & \tau' \end{bmatrix}, \quad u \in \mathbb{C}, \ \tau \in \mathrm{Mat}_{(n-1) \times (n-1)}(\mathbb{C})$$

and h ranges through a suitable lattice of $(n-1) \times (n-1)$ hermitian matrices. Once again, the coefficients $\mathbf{c}(u, v, w; h)$ vanish unless h is non-negative.

2.4. **Restrictions of automorphic forms.** Let us fix a choice of $K, V, \langle \cdot, \cdot \rangle$, the corresponding unitary group $U = U(V)$ of signature (m, n), and a choice of a congruence subgroup Γ as in the previous section. Consider a decomposition $V = V_1 \oplus V_2$ where V_1, V_2 are vector subspaces orthogonal for the pairing $\langle -, - \rangle$. This induces a natural embedding

$$\eta : U_1 \times U_2 \hookrightarrow U$$

where $U_i = U(V_i)$, $i = 1, 2$. Additionally, let us choose congruence subgroups $\Gamma_i \subseteq U_i(\mathbb{Q})$ such that $\eta(\Gamma_1 \times \Gamma_2) \subseteq \Gamma$. Let (m_i, n_i) be the corresponding signatures, so that $(m_1, n_1) + (m_2, n_2) = (m, n)$.

We fix a choice of one of the two versions of coordinates discussed in Sections 2.1 and 2.2, resp., the same for all groups U, U_1 and U_2. Explicitly, we consider one of the following two options:

(1) In the case (UB), we identify $U(\mathbb{R})$ with $U(m, n)$, $U_1(\mathbb{R})$ with $U(m_1, n_1)$ and $U_2(\mathbb{R})$ with $U(m_2, n_2)$. In this case, we set $\mathcal{H} = \mathcal{H}_{m,n}$, $\mathcal{H}^{(1)} = \mathcal{H}_{m_1,n_1}$ and $\mathcal{H}^{(2)} = \mathcal{H}_{m_2,n_2}$.
(2) When $m = n$, $m_1 = n_1$ and $m_2 = n_2$, with all the groups U_1, U_2, U quasi-split over $\mathbb{Q}$, we may consider the case (UT), i.e. we identify $U(\mathbb{R})$ with $U(\eta_n)$ and $U_i(\mathbb{R})$ with $U(\eta_{n_i})$, $i = 1, 2$. In this case, we set $\mathcal{H} = \mathcal{H}_n$, $\mathcal{H}^{(1)} = \mathcal{H}_{n_1}$ and $\mathcal{H}^{(2)} = \mathcal{H}_{n_2}$.

We will discuss both of these cases at once, in a unified way. To that end, from now on we use the term "automorphic form" to refer both to automorphic forms in the sense of Definition 2.2 in the case (UB), as well as to hermitian modular forms (from Definitiion 2.5) in the case (UT).

Regardless of which case occurs, the bases of $(V_i)_{\mathbb{R}}$ giving the chosen coordinates may be chosen so that η becomes the map

$$\eta : U_1(\mathbb{R}) \times U_2(\mathbb{R}) \hookrightarrow U(\mathbb{R}), \quad \left(\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}, \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} \right) \mapsto \begin{bmatrix} a_1 & & b_1 & \\ & a_2 & & b_2 \\ c_1 & & d_1 & \\ & c_2 & & d_2 \end{bmatrix}.$$

The corresponding embedding of symmetric spaces is then given by the map

$$\iota : \mathcal{H}^{(1)} \times \mathcal{H}^{(2)} \hookrightarrow \mathcal{H}, \quad (\tau_1, \tau_2) \mapsto \begin{bmatrix} \tau_1 & \\ & \tau_2 \end{bmatrix},$$

whcih is clearly $U_1(\mathbb{R}) \times U_2(\mathbb{R})$-equivariant in the obvious sense.

We fix a notation for coordinates on $\mathcal{H}$ compatibly with the embeddings, that is,

$$(3) \qquad \mathcal{H} \ni \tau = \begin{bmatrix} \tau_1 & x \\ y & \tau_2 \end{bmatrix}, \quad \tau_1 \in \mathcal{H}^{(1)}, \ \tau_2 \in \mathcal{H}^{(2)},$$

where $x = (x_{ij})$ and $y = (y_{ji})$ are rectangular blocks whose dimensions are determined by the blocks τ_1, τ_2. If needed, we will refer to the coordinates (x_{ij}) as "x-coordinates", and similarly, to (y_{ji}) as "y-coordinates".

Observe that for $\gamma = \eta(\gamma_1, \gamma_2) \in \Gamma$ with $\gamma_i \in \Gamma_i$ and $\tau = \iota(\tau_1, \tau_2)$ with $\tau_i \in \mathcal{H}^{(i)}$, we have

$$\lambda_\gamma(\tau) = \begin{bmatrix} \lambda_{\gamma_1}(\tau_1) & \\ & \lambda_{\gamma_2}(\tau_2) \end{bmatrix}, \quad \mu_\gamma(\tau) = \begin{bmatrix} \mu_{\gamma_1}(\tau_1) & \\ & \mu_{\gamma_2}(\tau_2) \end{bmatrix}.$$

It follows that, when k, l are arbitrary integers, the restriction of a form $f \in M_{(k,l)}(\Gamma)$ to a function on $\mathcal{H}^{(1)} \times \mathcal{H}^{(2)}$ defines a map

$$(4) \qquad M_{(k,l)}(\Gamma) \to M_{(k,l)}(\Gamma_1) \otimes M_{(k,l)}(\Gamma_2).$$

In terms of the coordinates (3), the map is given by $f \mapsto f|_{\substack{x=0 \\ y=0}}$, i.e. by restriction to the locus where all $x_{ij} = y_{ji} = 0$.

2.5. **The construction.** Fix the notation for $(k,l), \mathrm{U}_i, \mathrm{U}, \mathcal{H}^{(i)}, \Gamma_i, \Gamma$ etc. as in Section 2.4. Our next goal is to describe the desired differential operators on automorphic forms in coordinates. To formulate the construction, the following calculus notation will be useful.

Notation 2.9. Given a function $f(x_1, \ldots, x_l)$, and an (ordered) tuple of indicies $\alpha = (i_1, i_2, \ldots, i_r)$, denote

$$\partial_{x_\alpha} f = \frac{\partial^r f}{\partial x_{i_1} \partial x_{i_2} \ldots \partial x_{i_r}}.$$

For $r \geq 1$, $\mathrm{d}^r f$ denotes the r-th total differential of f, that is, the (symmetric) r-linear form on the tangent space given in coordinates as

$$\mathrm{d}^r f = \sum_\alpha \partial_{x_\alpha} f \mathrm{d} x_\alpha,$$

where $\alpha = (i_1, i_2, \ldots i_r)$ runs over all (ordered) r-tuples of indices for coordinates, and $\mathrm{d} x_\alpha$ denotes $\mathrm{d} x_{i_1} \mathrm{d} x_{i_2} \ldots \mathrm{d} x_{i_r}$.

When $g = (g_1, g_2, \ldots g_l)$ is a vector function in variables $y_1, y_2, \ldots, y_t$, we denote by $\mathrm{d}^r g$ the tensor $(\mathrm{d}^r g_1, \mathrm{d}^r g_2, \ldots \mathrm{d}^r g_l)$ (so that $\mathrm{d} g = \mathrm{d}^1 g$ agrees with the usual meaning of the tangent map).

Remark 2.10. In this notation, we have the following convenient forms of the chain rule:

$$\mathrm{d}(f \circ g) = \mathrm{d} f \circ \mathrm{d} g,$$

$$\mathrm{d}^r(f \circ g) = \sum_{a=1}^{r} \sum_{b_1 + b_2 + \cdots + b_a = r} \mathrm{d}^a f \circ (\mathrm{d}^{b_1} g, \mathrm{d}^{b_2} g, \ldots, \mathrm{d}^{b_a} g).$$

We are now ready to proceed with the construction. To make the construction more transparent, we start by formulating the case of the first derivative separately.

For every pair of non-negative integers (m, n), define

$$\rho^+_{(m,n),(k,l)} := \Delta_{(m,n),(k,l)} \otimes (\rho_{\mathrm{std}} \boxtimes \mathbf{1}) : \mathrm{GL}_m(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C}) \to \mathrm{GL}(\mathbb{C}^m \otimes \mathbb{C}) \simeq \mathrm{GL}_m(\mathbb{C}),$$

$$(U, V) \mapsto \det(U)^k \det(V)^l U,$$

$$\rho^-_{(m,n),(k,l)} = \Delta_{(m,n),(k,l)} \otimes (\mathbf{1} \boxtimes \rho_{\mathrm{std}}) : \mathrm{GL}_m(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C}) \to \mathrm{GL}(\mathbb{C} \otimes \mathbb{C}^n) \simeq \mathrm{GL}_n(\mathbb{C}),$$

$$(U, V) \mapsto \det(U)^k \det(V)^l V,$$

as representations of $\mathrm{GL}_s(\mathbb{C}) \times \mathrm{GL}_t(\mathbb{C})$. Here, $\Delta_{(m,n),(k,l)}$ is as in Example 2.4 and ρ_{std} (resp. $\mathbf{1}$) denotes the standard (resp. trivial) representation of the appropriate dimension.

Remark 2.11. When $(m,n) = (m_i, n_i)$ for $i = 1$ or 2, as in Section 2.4, we simply write $\rho^{\pm}_{i,(k,l)}$ for $\rho^{\pm}_{(m_i,n_i),(k,l)}$.

For a holomorphic function $f : \mathcal{H} \to \mathbb{C}$, we denote by $\mathrm{d}_x f|_{\mathcal{H}^{(1)} \times \mathcal{H}^{(2)}}$ the form

$$\mathrm{d}_x f|_{\mathcal{H}^{(1)} \times \mathcal{H}^{(2)}} = \mathrm{d}_x f|_{\substack{x=0 \\ y=0}} = \sum_{i,j} \frac{\partial f}{\partial x_{ij}} \mathrm{d}x_{ij}\Bigg|_{\substack{x_{ij}=0 \\ y_{ji}=0}},$$

that is, the form obtained from $\mathrm{d}f$ by projection onto the span of differentials of the x-coordinates, and then setting all x- and y-coordinates to 0.

Similarly, $\mathrm{d}_y f|_{\mathcal{H}^{(1)} \times \mathcal{H}^{(2)}}$ is defined as

$$\mathrm{d}_y f|_{\mathcal{H}^{(1)} \times \mathcal{H}^{(2)}} = \mathrm{d}_y f|_{\substack{x=0 \\ y=0}} = \sum_{j,i} \frac{\partial f}{\partial y_{ji}} \mathrm{d}y_{ji}\Bigg|_{\substack{x_{ij}=0 \\ y_{ji}=0}},$$

i.e. the analogous form where we project $\mathrm{d}f$ onto $\mathrm{d}y$-coordinates instead before restricting to $\mathcal{H}^{(1)} \times \mathcal{H}^{(2)}$.

Proposition 2.12. *Let $f : \mathcal{H} \to \mathbb{C}$ be an automorphic form of weight (k, l), and assume that the restriction $f|_{\mathcal{H}^{(1)} \times \mathcal{H}^{(2)}}$ vanishes.*

Then, the differential form $\mathrm{d}_x f|_{\mathcal{H}^{(1)} \times \mathcal{H}^{(2)}}$ is a tensor product of vector-valued automorphic forms of level Γ_1 and Γ_2 respectively, and weight $\rho^+_{1,(k,l)}$ and $\rho^-_{2,(k,l)}$ respectively.

Similarly, the form $\mathrm{d}_y f|_{\mathcal{H}^{(1)} \times \mathcal{H}^{(2)}}$ is a tensor product of vector-valued automorphic forms of level Γ_1 and Γ_2 respectively, and weight $\rho^-_{1,(k,l)}$ and $\rho^+_{2,(k,l)}$ respectively.

Remark 2.13. To better understand the content of Proposition 2.12, let us be more explicit about the expected modularity rule. By convention, we identify the space $\mathrm{Mat}_{m_1 \times n_2}(\mathbb{C})$ of $m_1 \times n_2$ complex matrices with the $\mathrm{GL}_{m_1}(\mathbb{C}) \times \mathrm{GL}_{n_2}(\mathbb{C})$-representation $\rho_{\mathrm{std}} \boxtimes \rho_{\mathrm{std}}$, with the action given by the formula

$$\rho(A, B)(X) = AX\,^t B, \quad (A, B) \in \mathrm{GL}_{m_1}(\mathbb{C}) \times \mathrm{GL}_{n_2}(\mathbb{C}), \quad X \in \mathrm{Mat}_{m_1 \times n_2}(\mathbb{C}).$$

The form $\mathrm{d}_x f|_{\substack{x=0 \\ y=0}}$ naturally takes values in the dual space $\mathrm{Mat}_{m_1 \times n_2}(\mathbb{C})^{\vee}$, on which $\mathrm{GL}_{m_1}(\mathbb{C}) \times \mathrm{GL}_{n_2}(\mathbb{C})$ acts via

$$\rho'(A, B)(\alpha)(X) = \alpha(\,^t A X B) \ \ (= \alpha(\rho(\,^t A, \,^t B)(X))),$$

and it is easy to see that the resulting action makes $\mathrm{Mat}_{m_1 \times n_2}(\mathbb{C})^{\vee}$ into a representation which is again isomorphic to $\rho_{\mathrm{std}} \boxtimes \rho_{\mathrm{std}}$.

The modularity condition of Proposition 2.12 can then be rephrased as follows: assuming the vanishing of $f|_{\mathcal{H}^{(1)} \times \mathcal{H}^{(2)}}$, for an element γ of the form

$$\gamma = \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} a_1 & & b_1 & \\ & a_2 & & b_2 \\ c_1 & & d_1 & \\ & c_2 & & d_2 \end{bmatrix} = \eta(\gamma_1, \gamma_2) \in \Gamma,$$

$$(\gamma_1, \gamma_2) = \left(\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}, \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} \right) \in \Gamma_1 \times \Gamma_2,$$

and $(\tau_1, \tau_2) \in \mathcal{H}^{(1)} \times \mathcal{H}^{(2)}$, we have

$$\mathrm{d}_x f \left(\begin{bmatrix} \tau_1 & \\ & \tau_2 \end{bmatrix} \right) =$$

$$= \rho'(\rho_{m_1,n_1}^+(\lambda_{\gamma_1}(\tau_1), \mu_{\gamma_1}(\tau_1))^{-1}, \rho_{m_2,n_2}^-(\lambda_{\gamma_2}(\tau_2), \mu_{\gamma_2}(\tau_2))^{-1}) \left(\mathrm{d}_x f \left(\begin{bmatrix} \gamma_1 \tau_1 & \\ & \gamma_2 \tau_2 \end{bmatrix} \right) \right).$$

Similar interpretation applies to the case of the form $\mathrm{d}_y f|_{\mathcal{H}^{(1)} \times \mathcal{H}^{(2)}}$ (where we instead consider the space $\mathrm{Mat}_{m_2 \times n_1}(\mathbb{C})$ etc.).

Let us now describe the general case of higher order derivatives. Given a holomorphic map $f : \mathcal{H} \to \mathbb{C}$, the form $\mathrm{d}^r f$ projected onto the $\mathrm{d}x$-coordinates and restricted to $\mathcal{H}^{(1)} \times \mathcal{H}^{(2)}$,

$$(5) \qquad \mathrm{d}_x^r f|_{\mathcal{H}^{(1)} \times \mathcal{H}^{(2)}} = \mathrm{d}_x^r f|_{\substack{x=0 \\ y=0}} = \sum_\alpha \partial_{x_\alpha} f|_{\substack{x=0 \\ y=0}} \mathrm{d}x_\alpha$$

(where α runs over all r-tuples of indices for the x-coordinates), is a map from $\mathcal{H}^{(1)} \times \mathcal{H}^{(2)}$ that naturally lands in the space of r-linear forms on $m_1 \times n_2$ complex matrices, that is, in $\mathrm{Sym}^r(\mathrm{Mat}_{m_1 \times n_2}(\mathbb{C})^\vee)$.

The natural left action of $\mathrm{GL}_{m_1}(\mathbb{C}) \times \mathrm{GL}_{n_2}(\mathbb{C})$ on $\mathrm{Mat}_{m_1 \times n_2}(\mathbb{C})$ (as outlined in Remark 2.13) gives a right action $*$ on $\mathrm{Sym}^r(\mathrm{Mat}_{m_1 \times n_2}(\mathbb{C})^\vee)$ via

$$[\beta * (A, B)](X_1 \dots X_r) = \beta(A X_1 {}^t B, A X_2 {}^t B, \dots, A X_r {}^t B),$$

where $(A, B) \in \mathrm{GL}_{m_1}(\mathbb{C}) \times \mathrm{GL}_{n_2}(\mathbb{C})$, which we make into left action by the rule

$$(6) \qquad \rho'(A, B)(\beta) = \beta * ({}^t A, {}^t B).$$

On the collection of coefficients $(\partial_{x_\alpha} f|_{\substack{x=0 \\ y=0}})_\alpha$, the corresponding action is the expected action $\mathrm{Sym}^r(\rho_{\mathrm{std}} \boxtimes \rho_{\mathrm{std}})$ where ρ_{std} again stands for the standard representation of GL_{m_1} and GL_{n_2}, respectively.

We have the decomposition

$$(7) \qquad \mathrm{Sym}^r(\rho_{\mathrm{std}} \boxtimes \rho_{\mathrm{std}}) = \bigoplus_\lambda \mathbb{S}^\lambda(\rho_{\mathrm{std}}) \boxtimes \mathbb{S}^\lambda(\rho_{\mathrm{std}})$$

(e.g. by [FH13, Exercises 6.11(b), 4.51(b)] or [Wey03, Corollary 2.3.3]) where λ runs over partitions of r and $\mathbb{S}^\lambda$ denotes the Schur functor.

For each λ, denote by

$$\mathrm{d}_{x,\lambda}^r f|_{\mathcal{H}^{(1)} \times \mathcal{H}^{(2)}} = \mathrm{d}_{x,\lambda}^r f\Big|_{\substack{x=0 \\ y=0}}$$

the map $\mathrm{d}_x^r f|_{\mathcal{H}^{(1)} \times \mathcal{H}^{(2)}}$ composed with the projection onto the $\mathbb{S}^\lambda \boxtimes \mathbb{S}^\lambda$-factor.

By a similar discussion, we define the forms $\mathrm{d}_{y,\lambda}^r f|_{\mathcal{H}^{(1)} \times \mathcal{H}^{(2)}}$ (the only difference being the dimensions of the matrix space, which is in this case $\mathrm{Mat}_{m_1 \times n_2}(\mathbb{C})$). The higher-derivative analogue of Proposition 2.12 is the following.

Theorem 2.14. *As in the situation of Proposition 2.12, assume that the forms $\mathrm{d}_x^s f|_{\mathcal{H}^{(1)} \times \mathcal{H}^{(2)}}$ vanish for all s with $0 \le s < r$. Then for every partition $\lambda \vdash r$, the*

form $\left.\mathrm{d}_{x,\lambda}^r f\right|_{\mathcal{H}^{(1)} \times \mathcal{H}^{(2)}}$ is a tensor product of automorphic forms of weights

$$\rho_{1,(k,l)}^{+,\lambda} := \Delta_{(m_1,n_1),(k,l)} \otimes (\mathbb{S}^\lambda(\rho_{\mathrm{std}}) \boxtimes \mathbf{1}) = \left(\det{}^k \otimes \mathbb{S}^\lambda(\rho_{\mathrm{std}})\right) \boxtimes \det{}^l$$

and

$$\rho_{2,(k,l)}^{-,\lambda} := \Delta_{(m_2,n_2),(k,l)} \otimes (\mathbf{1} \boxtimes \mathbb{S}^\lambda(\rho_{\mathrm{std}})) = \det{}^k \boxtimes \left(\det{}^l \otimes \mathbb{S}^\lambda(\rho_{\mathrm{std}})\right).$$

Similarly, if the forms $\left.\mathrm{d}_y^s f\right|_{\mathcal{H}^{(1)} \times \mathcal{H}^{(2)}}$ vanish for all $s < r$, then the form $\left.\mathrm{d}_{y,\lambda}^r f\right|_{\mathcal{H}^{(1)} \times \mathcal{H}^{(2)}}$ is a tensor product of automorphic forms of weights

$$\rho_{1,(k,l)}^{-,\lambda} := \Delta_{(m_1,n_1),(k,l)} \otimes (\mathbf{1} \boxtimes \mathbb{S}^\lambda(\rho_{\mathrm{std}})) = \det{}^k \boxtimes \left(\det{}^l \otimes \mathbb{S}^\lambda(\rho_{\mathrm{std}})\right)$$

and

$$\rho_{2,(k,l)}^{+,\lambda} := \Delta_{(m_2,n_2),(k,l)} \otimes (\mathbb{S}^\lambda(\rho_{\mathrm{std}}) \boxtimes \mathbf{1}) = \left(\det{}^k \otimes \mathbb{S}^\lambda(\rho_{\mathrm{std}})\right) \boxtimes \det{}^l.$$

Remark 2.15. By convention, we identify both $\mathrm{d}_x^0 f$ and $\mathrm{d}_y^0 f$ with f itself, so that vanishing of either of the forms $\left.\mathrm{d}_x^0 f\right|_{\mathcal{H}^{(1)} \times \mathcal{H}^{(2)}}$ or $\left.\mathrm{d}_y^0 f\right|_{\mathcal{H}^{(1)} \times \mathcal{H}^{(2)}}$ is equivalent to the assumption $f|_{\mathcal{H}^{(1)} \times \mathcal{H}^{(2)}} = 0$ of Proposition 2.12.

Remark 2.16. The choices of coordinates used in this section, i.e. for cases (UB) and (UT), are especially useful in describing our differential operators explicitly. However, it will be a consequence of the discussion in Section 5 that the construction described in this section is in fact coordinate-independent. In particular, the obvious statements for automorphic forms as described in Section 2.3 remain valid.

Let us describe one particular case where the different variants of coordinates will be useful later on. Suppose that $m \neq n$ and that the factor U_1 is quasi-split and of signature $(1,1)$. We may then consider the diagonal embedding of symmetric spaces $\mathcal{H}_1 \times \widetilde{\mathcal{H}}_{m-1,n-1} \hookrightarrow \widetilde{\mathcal{H}}_{m,n}$, and fix coordinates on $\widetilde{\mathcal{H}}_{m,n}$ accordingly, i.e.

$$(8) \qquad \widetilde{\mathcal{H}}_{m,n} \ni \begin{bmatrix} \tau \\ u \end{bmatrix} = \begin{bmatrix} \tau_1 & x \\ z & \tau_2 \\ w & u_2 \end{bmatrix}, \quad \tau_1 \in \mathcal{H}_1, \ \begin{bmatrix} \tau_2 \\ u_2 \end{bmatrix} \in \widetilde{\mathcal{H}}_{m-1,n-1}.$$

Letting y denote the column vector ${}^t[z \quad w]$, the operators $\left.\mathrm{d}_{x,\lambda}^r(-)\right|_{\mathcal{H}_1 \times \widetilde{\mathcal{H}}_{m-1,n-1}}$ and $\left.\mathrm{d}_{y,\lambda}^r(-)\right|_{\mathcal{H}_1 \times \widetilde{\mathcal{H}}_{m-1,n-1}}$ make sense and satisfy conclusions of Theorem 2.14 (producing Hermitian modular forms for the first factor and automorphic forms in the sense of Section 2.3 for the second factor).

3. Example: Restricting a Hermitian Analog of the Schottky Form

Hentschel and Krieg construct a Hermitian analog of the Schottky form as a suitable linear combination of Hermitian theta series of even unimodular Gaussian lattices [HK06]. We will briefly review their construction, and then use it to construct a vector-valued automorphic form.

The three Hermitian positive definite matrices

$$
\begin{bmatrix}
2 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\
0 & 2 & 0 & 0 & -1 & 1 & -1 & 0 \\
0 & 0 & 2 & 0 & 0 & 1 & 1 & -1 \\
0 & 0 & 0 & 2 & -1 & 0 & 1 & 1 \\
1 & -1 & 0 & -1 & 2 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 2 & 0 & 0 \\
0 & -1 & 1 & 1 & 0 & 0 & 2 & 0 \\
1 & 0 & -1 & 1 & 0 & 0 & 0 & 2
\end{bmatrix},
\quad
\begin{bmatrix}
2 & -1 & 0 & -1 & -1 & -1 & -i & 1+i \\
-1 & 2 & 1-i & 0 & 0 & 0 & 0 & -i \\
0 & 1+i & 2 & 0 & 0 & 0 & 0 & 1 \\
-1 & 0 & 0 & 2 & 1 & 1 & i & -1 \\
-1 & 0 & 0 & 1 & 2 & 1 & i & -1 \\
-1 & 0 & 0 & 1 & 1 & 2 & i & -1 \\
i & 0 & 0 & -i & -i & -i & 2 & i \\
1-i & i & 1 & -1 & -1 & -1 & -i & 4
\end{bmatrix},
\quad
\begin{bmatrix}
2 & 0 & 1+i & i & 0 & 0 & 0 & 0 \\
0 & 2 & i & 1-i & 0 & 0 & 0 & 0 \\
1-i & -i & 2 & 0 & 0 & 0 & 0 & 0 \\
-i & 1+i & 0 & 2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 2 & 0 & 1+i & i \\
0 & 0 & 0 & 0 & 0 & 2 & i & 1-i \\
0 & 0 & 0 & 0 & 1-i & -i & 2 & 0 \\
0 & 0 & 0 & 0 & -i & 1+i & 0 & 2
\end{bmatrix}
$$

will be denoted by S_1, S_2, and S_3, respectively. Their exact values will not be important to this discussion, but they arise as the Gram matrices of the three isometry classes of even unimodular Gaussian lattices of rank 8 [HK06]. Each Hermitian positive definite matrix S_i gives rise to the Hermitian inner product $\langle v, w \rangle_i = w^* S_i v$ on $\mathbb{Z}[i]^8$.

We can then define Hermitian theta series

$$
\Theta_i^{(n)}(\tau) = \sum_{M \in \mathbb{Z}[i]^{8 \times n}} \exp(\pi i \operatorname{tr}({}^t \overline{M} S_i M \tau)) = \sum_h a_i^{(n)}(h) \exp(2\pi i \operatorname{tr}(h\tau))
$$

with Fourier coefficients

$$
a_i^{(n)}(h) = \#\{M \in \mathbb{Z}[i]^{8 \times n} : {}^t\overline{M} S_i M = 2h\}
$$

(9)
$$
= \#\{v_1, \ldots, v_n \in \mathbb{Z}[i]^8 : \langle v_j, v_k \rangle_i = 2h_{kj}\}.
$$

Hentschel and Krieg consider the linear combination $F^{(n)} = 8\Theta_1^{(n)} - 15\Theta_2^{(n)} + 7\Theta_3^{(n)}$ and demonstrate that $F^{(4)}$ is a Hermitian analog of the Schottky form [HK06].

Lemma 3.1. *The linear combination $F^{(4)} = 8\Theta_1^{(4)} - 15\Theta_2^{(4)} + 7\Theta_3^{(4)}$ is a nonzero cusp form of weight 8, and the restriction $F^{(4)}|_{\mathcal{S}_4}$ is a multiple of the Schottky form.*

Proof. This is [HK06, Theorem 3.1(c) and Corollary 3.4]. $\square$

In contrast, $F^{(1)}$, $F^{(2)}$, and $F^{(3)}$ all vanish.

Lemma 3.2. *The linear combinations $F^{(n)} = 8\Theta_1^{(n)} - 15\Theta_2^{(n)} + 7\Theta_3^{(n)}$ vanish for $n \le 3$. For $n = 1$, we have $\Theta_1^{(1)} = \Theta_2^{(1)} = \Theta_3^{(1)}$.*

Proof. From the combinatorial description of $a_i^{(n)}(h)$ given in Equation (9), we have

$$
a_i^{(n)}(h) = a_i^{(n+1)}\left(\begin{bmatrix} h & \\ & 0 \end{bmatrix}\right).
$$

In other words, each Fourier coefficient of $\Theta_i^{(n)}$ appears as a singular Fourier coefficient of $\Theta_i^{(n+1)}$. The same is true for the linear combination $F^{(n)}$. In particular, if all singular Fourier coefficients of $F^{(n+1)}$ vanish (i.e., if $F^{(n+1)}$ is a cusp form), then $F^{(n)}$ must vanish. Since $F^{(4)}$ is a cusp form, this shows that $F^{(n)}$ vanishes for $n \le 3$.

Lemma 3.3(c) in [HK06] states that $\Theta_1^{(n)}|_{\mathcal{S}_n} = \Theta_3^{(n)}|_{\mathcal{S}_n}$, where $\mathcal{S}_n$ denotes the Siegel upper half-space of degree n. In particular, we have $\Theta_1^{(1)} = \Theta_3^{(1)}$ since $\mathcal{S}_1 = \mathcal{H}_1$. But then the relation $8\Theta_1^{(1)} - 15\Theta_2^{(1)} + 7\Theta_3^{(1)} = 0$ forces $\Theta_1^{(1)} = \Theta_2^{(1)} = \Theta_3^{(1)}$. $\square$

Remark 3.3. Let us be explicit about the setup for our example. We fix the field $K = \mathbb{Q}(\sqrt{-1}) = \mathbb{Q}(i)$ and consider the groups $U(\eta_n)$ treated as algebraic groups with the obvious integral model $\mathcal{U}(\eta_n)$, i.e.

$$\mathcal{U}(\eta_n)(A) = \{g \in \mathrm{GL}_{2n}(A \otimes_{\mathbb{Z}} \mathbb{Z}[i]) \mid {}^t\overline{g}\eta_n g = \eta_n\}, \quad A \in \mathsf{Alg}_{\mathbb{Z}}.$$

The form $F^{(4)}$ is then Hermitian modular of weight $(0,8)$ and full level $\Gamma = \mathcal{U}(\eta_4)(\mathbb{Z})$.

To apply our construction, we consider the diagonal embedding of $\mathcal{U}(\eta_3) \times \mathcal{U}(\eta_1)$ into $\mathcal{U}(\eta_4)$. It is worth noting that the standard representation ρ_{std} associated with the first factor as in Section 2.5 is one-dimensional. Consequently, the decomposition (Equation (7)) of $\mathrm{Sym}^r(\rho_{\mathrm{std}} \boxtimes \rho_{\mathrm{std}})$ in terms of Schur functors is trivial, i.e. the $\mathbb{S}^\lambda \boxtimes \mathbb{S}^\lambda$ terms will vanish unless $\lambda = (r)$. That is, we have $\mathrm{Sym}^r(\rho_{\mathrm{std}} \boxtimes \rho_{\mathrm{std}}) \simeq \mathrm{Sym}^r(\rho_{\mathrm{std}}) \boxtimes \mathrm{Sym}^r(\rho_{\mathrm{std}})$, and there is no need to take any projection to a $\mathbb{S}^\lambda \boxtimes \mathbb{S}^\lambda$-component in our construction.

We will show that $F^{(4)}$ vanishes to order 4 along $\mathcal{H}_3 \times \mathcal{H}_1$ and that the fourth derivative $\mathrm{d}_x^4 F^{(4)}|_{\mathcal{H}_3 \times \mathcal{H}_1}$ is a nonzero vector-valued automorphic cusp form. We will do this by explicitly computing the Fourier expansion of the derivatives $\mathrm{d}_x^r F^{(4)}|_{\mathcal{H}_3 \times \mathcal{H}_1}$ and making use of the combinatorial description of $a_i^{(n)}(h)$ given in Equation (9).

Remark 3.4. Using the notation of Section 2.1, the weight of $F^{(4)}$ is the 1-dimensional representation $\Delta_{0,8} = \mathbf{1} \boxtimes \det^8$ on $\mathrm{GL}_4(\mathbb{C}) \times \mathrm{GL}_4(\mathbb{C})$. Upon restriction to $\mathcal{U}(\eta_3) \times \mathcal{U}(\eta_1)$, its weight is $(\mathbf{1} \boxtimes \det^8) \boxtimes (\det^8 \boxtimes \mathbf{1})$ on $(\mathrm{GL}_3(\mathbb{C}) \times \mathrm{GL}_3(\mathbb{C})) \times (\mathrm{GL}_1(\mathbb{C}) \times \mathrm{GL}_1(\mathbb{C}))$.

The representation on $\mathrm{GL}_1(\mathbb{C}) \times \mathrm{GL}_1(\mathbb{C})$ corresponds to the weight of modular forms on $\mathcal{U}(\eta_1) \cong \mathrm{SL}_2(\mathbb{C})$; in this case, modular forms of weight 8 on $\mathrm{SL}_2(\mathbb{C})$.

Lastly, according to Theorem 2.14 (omitting any choice of partitions λ of $r = 4$), the weight of $\mathrm{d}_x^4 F^{(4)}|_{\mathcal{H}_3 \times \mathcal{H}_1}$ and $d_y^4 F^{(4)}|_{\mathcal{H}_3 \times \mathcal{H}_1}$ are the representations

$$(10) \qquad\qquad (\mathrm{Sym}^4(\rho_{\mathrm{std}}) \boxtimes \det^8) \boxtimes (\det^8 \boxtimes \det^4)$$

and

$$(11) \qquad\qquad (\mathbf{1} \boxtimes (\det^8 \otimes \mathrm{Sym}^4(\rho_{\mathrm{std}}))) \boxtimes (\det^{12} \boxtimes \mathbf{1})$$

of $\mathrm{GL}_3(\mathbb{C}) \times \mathrm{GL}_3(\mathbb{C}) \times \mathrm{GL}_1(\mathbb{C}) \times \mathrm{GL}_1(\mathbb{C})$ respectively. Once more, the $\mathrm{GL}_1(\mathbb{C}) \times \mathrm{GL}_1(\mathbb{C})$ part corresponds to the weight of modular forms on $\mathcal{U}(\eta_1) \cong \mathrm{SL}_2(\mathbb{C})$. In both cases, we obtain modular forms of weight 12 on $\mathrm{SL}_2(\mathbb{C})$.

Set $c_1 = 8$, $c_2 = -15$, and $c_3 = 7$. Let $\tau_1 \in \mathcal{H}_3$ and $\tau_2 \in \mathcal{H}_1$. Then the Fourier expansion of $F^{(4)}|_{\mathcal{H}_3 \times \mathcal{H}_1}$ is given by

$$F^{(4)}\left(\begin{bmatrix} \tau_1 & \\ & \tau_2 \end{bmatrix}\right) = \sum_i c_i \Theta_i^{(4)}\left(\begin{bmatrix} \tau_1 & \\ & \tau_2 \end{bmatrix}\right)$$

$$= \sum_i c_i \sum_{h_1,h_2,h_3} a_i^{(4)}\left(\begin{bmatrix} h_1 & {}^t\overline{h_3} \\ h_3 & h_2 \end{bmatrix}\right) \exp\left(2\pi i \, \mathrm{tr}\left(\begin{bmatrix} h_1 & {}^t\overline{h_3} \\ h_3 & h_2 \end{bmatrix}\begin{bmatrix} \tau_1 & \\ & \tau_2 \end{bmatrix}\right)\right)$$

$$(12) \qquad = \sum_i c_i \sum_{h_1,h_2} \exp\left(2\pi i \, \mathrm{tr}(h_1\tau_1)\right) \exp(2\pi i \, \mathrm{tr}(h_2\tau_2)) \sum_{h_3} a_i^{(4)}\left(\begin{bmatrix} h_1 & {}^t\overline{h_3} \\ h_3 & h_2 \end{bmatrix}\right).$$

More generally, we can compute

$$F^{(4)}\left(\begin{bmatrix} \tau_1 & x \\ y & \tau_2 \end{bmatrix}\right) = \sum_i c_i \Theta_i^{(4)}\left(\begin{bmatrix} \tau_1 & x \\ y & \tau_2 \end{bmatrix}\right)$$

$$= \sum_i c_i \sum_{h_1, h_2, h_3} a_i^{(4)}\left(\begin{bmatrix} h_1 & {}^t\overline{h_3} \\ h_3 & h_2 \end{bmatrix}\right) \exp\left(2\pi i \operatorname{tr}\left(\begin{bmatrix} h_1 & {}^t\overline{h_3} \\ h_3 & h_2 \end{bmatrix}\begin{bmatrix} \tau_1 & x \\ y & \tau_2 \end{bmatrix}\right)\right)$$

$$= \sum_i c_i \sum_{h_1, h_2, h_3} a_i^{(4)}\left(\begin{bmatrix} h_1 & {}^t\overline{h_3} \\ h_3 & h_2 \end{bmatrix}\right) \exp(2\pi i \operatorname{tr}(h_1\tau_1 + h_2\tau_2 + h_3 x + {}^t\overline{h_3}y)).$$

Then the Fourier expansion of $\mathrm{d}_x^r F^{(4)}|_{\mathcal{H}_3 \times \mathcal{H}_1}$ is given by

$$\mathrm{d}_x^r F^{(4)}\left(\begin{bmatrix} \tau_1 & \\ & \tau_2 \end{bmatrix}\right)$$

$$= \sum_\alpha \sum_i c_i \sum_{h_1, h_2, h_3} a_i^{(4)}\left(\begin{bmatrix} h_1 & {}^t\overline{h_3} \\ h_3 & h_2 \end{bmatrix}\right) (2\pi i)^r h_3^\alpha \exp(2\pi i \operatorname{tr}(h_1\tau_1 + h_2\tau_2)) \, \mathrm{d}x_\alpha$$

(13)

$$= (2\pi i)^r \sum_\alpha \sum_{h_1, h_2} \exp(2\pi i \operatorname{tr}(h_1\tau_1)) \exp(2\pi i \operatorname{tr}(h_2\tau_2)) \sum_i c_i \sum_{h_3} h_3^\alpha a_i^{(4)}\left(\begin{bmatrix} h_1 & {}^t\overline{h_3} \\ h_3 & h_2 \end{bmatrix}\right) \, \mathrm{d}x_\alpha.$$

Proposition 3.5. *The restriction $F^{(4)}|_{\mathcal{H}_3 \times \mathcal{H}_1}$ vanishes.*

Proof. The combinatorial description of $a_i^{(n)}(h)$ given in Equation (9) tells us that

$$\sum_{h_3} a_i^{(4)}\left(\begin{bmatrix} h_1 & {}^t\overline{h_3} \\ h_3 & h_2 \end{bmatrix}\right) = a_i^{(3)}(h_1) a_i^{(1)}(h_2).$$

Then Equation (12) and Lemma 3.2 give

$$F^{(4)}\left(\begin{bmatrix} \tau_1 & \\ & \tau_2 \end{bmatrix}\right) = \sum_i c_i \Theta_i^{(3)}(\tau_1) \Theta_i^{(1)}(\tau_2) = F^{(3)}(\tau_1) \Theta^{(1)}(\tau_2) = 0. \qquad \square$$

Proposition 3.6. *The restrictions $\mathrm{d}_x^r F^{(4)}|_{\mathcal{H}_3 \times \mathcal{H}_1}$ vanish for $r \not\equiv 0 \pmod 4$, but the restriction $\mathrm{d}_x^4 F^{(4)}|_{\mathcal{H}_3 \times \mathcal{H}_1}$ does not vanish.*

Proof. Equation (13) tells us that each Fourier coefficient of $\mathrm{d}_x^r F^{(4)}|_{\mathcal{H}_3 \times \mathcal{H}_1}$ is of the form

$$(2\pi i)^r \sum_i c_i \sum_{h_3} h_3^\alpha a_i^{(4)}\left(\begin{bmatrix} h_1 & {}^t\overline{h_3} \\ h_3 & h_2 \end{bmatrix}\right)$$

for fixed h_1, h_2, and α. Equation (9) lets us rewrite this as

$$(2\pi i)^r \sum_i c_i \sum_{\substack{v_1, v_2, v_3 \\ \langle v_j, v_k \rangle_i = 2(h_1)_{kj}}} \sum_{\substack{v_4 \\ \langle v_4, v_4 \rangle_i = 2h_2}} \langle v_1, v_4 \rangle_i^{\alpha_1} \langle v_2, v_4 \rangle_i^{\alpha_2} \langle v_3, v_4 \rangle_i^{\alpha_3}$$

where $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ with $\alpha_1 + \alpha_2 + \alpha_3 = r$.

If $r \not\equiv 0 \pmod 4$, then the values of the inner sum at v_4, iv_4, $-v_4$, and $-iv_4$ will cancel with each other, so every Fourier coefficient of $\mathrm{d}_x^r F^{(4)}|_{\mathcal{H}_3 \times \mathcal{H}_1}$ vanishes.

To show that the restriction $\mathrm{d}_x^4 F^{(4)}|_{\mathcal{H}_3 \times \mathcal{H}_1}$ does not vanish, it is enough to find one Fourier coefficient that does not vanish. Set $h_1 = I_3$, $h_2 = 1$, and $\alpha = (4, 0, 0)$. Then the Fourier coefficient in question is given by

$$(2\pi i)^4 \sum_i c_i \sum_{\substack{v_1, v_2, v_3 \\ \langle v_j, v_k \rangle_i = 2\delta_{jk}}} \sum_{\substack{v_4 \\ \langle v_4, v_4 \rangle_i = 2}} \langle v_1, v_4 \rangle^4.$$

For each i, the number of vectors v satisfying $\langle v, v \rangle_i = 2$ is exactly 480. For each i, let $\mathcal{C}_i$ denote the set of these 480 vectors. Then we can write the sum as

$$\sum_i c_i \sum_{v_1 \in \mathcal{C}_i} \left[\sum_{\substack{v_2, v_3 \in \mathcal{C}_i \\ \langle v_j, v_k \rangle_i = \delta_{jk}}} 1 \right] \left[\sum_{v_4 \in \mathcal{C}_i} \langle v_1, v_4 \rangle^4 \right]$$

which we can compute to be exactly 1981808640.

In order to enumerate the 480 elements of each $\mathcal{C}_i$, we found it helpful to use the Cholesky decomposition $S_i = d_i^{-1} \, {}^t\overline{L_i} D_i L_i$, so that the problem of finding ${}^t\overline{v} S_i v = 2$ becomes the simpler problem of finding ${}^t\overline{L_i v} D_i (L_i v) = 2d_i$. $\qquad\square$

Theorem 3.7. *The restriction $\mathrm{d}_x^4 F^{(4)}|_{\mathcal{H}_3 \times \mathcal{H}_1}$ is a nonzero vector-valued automorphic form. It can be written as a pure tensor $M \otimes \Delta$.*

Proof. Theorem 2.14 and Remark 3.3 show that if the restrictions $\mathrm{d}_x^s F^{(4)}|_{\mathcal{H}_3 \times \mathcal{H}_1}$ vanish for all $s < r$, then the restriction $\mathrm{d}_x^r F^{(4)}|_{\mathcal{H}_3 \times \mathcal{H}_1}$ is a vector-valued automorphic form. Then Propositions 3.5 and 3.6 tell us that $\mathrm{d}_x^4 F^{(4)}|_{\mathcal{H}_3 \times \mathcal{H}_1}$ is a nonzero vector-valued automorphic form. It is a tensor product of vector-valued automorphic forms for $\mathcal{U}(\eta_3)$ and scalar-valued automorphic forms for $\mathcal{U}(\eta_1)$ of weight 12. Then we can write $\mathrm{d}_x^4 F^{(4)}|_{\mathcal{H}_3 \times \mathcal{H}_1} = M_0 \otimes E_{12} + M \otimes \Delta$. But comparing Fourier expansions with Equation (13) forces $M_0 = 0$ and $\mathrm{d}_x^4 F^{(4)}|_{\mathcal{H}_3 \times \mathcal{H}_1} = M \otimes \Delta$. $\qquad\square$

In contrast, the restriction $F^{(4)}|_{\mathcal{H}_2 \times \mathcal{H}_2}$ does not vanish.

Proposition 3.8. *The restriction $F^{(4)}|_{\mathcal{H}_2 \times \mathcal{H}_2}$ does not vanish.*

Proof. Recall from Lemma 3.2 that the theta series $\Theta_i^{(2)}$ satisfy the linear relation $8\Theta_1^{(2)} - 15\Theta_2^{(2)} + 7\Theta_3^{(2)} = 0$. There are no further relations since the theta series $\Theta_i^{(2)}$ span a vector space of dimension 2. One way to see this is to observe that the theta series $\Theta_i^{(2)}$ are not cusp forms, but [HK06, Theorem 3.1(b)] states that the linear combination $-8\Theta_1^{(2)} + 3\Theta_2^{(2)} + 5\Theta_3^{(2)}$ is a nonzero cusp form.

Now suppose that the restriction $F^{(4)}|_{\mathcal{H}_2 \times \mathcal{H}_2}$ did vanish. Then, as in the proof of Proposition 3.5, we would have have

$$F^{(4)}\left(\begin{bmatrix} \tau_1 & \\ & \tau_2 \end{bmatrix}\right) = 8\Theta_1^{(2)}(\tau_1)\Theta_1^{(2)}(\tau_2) - 15\Theta_2^{(2)}(\tau_1)\Theta_2^{(2)}(\tau_2) + 7\Theta_3^{(2)}(\tau_1)\Theta_3^{(2)}(\tau_2) = 0.$$

For each fixed τ_2, this is a linear relation on the functions $\Theta_i^{(2)}(\tau_1)$. This relation must be a multiple of the relation $8\Theta_1^{(2)} - 15\Theta_2^{(2)} + 7\Theta_3^{(2)} = 0$. But this would require $\Theta_1^{(2)}(\tau_2) = \Theta_2^{(2)}(\tau_2) = \Theta_3^{(2)}(\tau_2)$ for all τ_2, which is false. $\qquad\square$

4. Proofs of Main Results

We now prove the main assertion, Theorem 2.14 (as well as Proposition 2.12, which is a special case). Fix all the notation $((k,l), U_i, U, \mathcal{H}^{(i)}, \Gamma_i, \Gamma \ldots$ etc.) as in Sections 2.4 and 2.5.

4.1. **Modularity.** Firstly, we prove that the functions resulting from our construction obey the expected modularity rules. The key ingredient for this part of the proof is the following lemma on the differential of the action of Γ on $\mathcal{H}$.

Lemma 4.1 ([Shi00, Lemma 3.4]).

$$d(\gamma\tau) = {}^t\lambda_\gamma(\tau)^{-1} d\tau \mu_\gamma(\tau)^{-1}, \quad \gamma = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in U(\mathbb{R}), \quad \tau \in \mathcal{H}.$$

Corollary 4.2. *Let* $\gamma = \eta(\gamma_1, \gamma_2)$ *be as in Remark 2.13. Then for every* $s > 0$, $d_x^s(\gamma\tau)|_{\substack{x=0 \\ y=0}}$ *is of the form*

$$d_x^s(\gamma\tau)|_{\substack{x=0 \\ y=0}} = \begin{bmatrix} 0 & * \\ 0 & 0 \end{bmatrix},$$

that is, it is a matrix of symmetric forms with all forms outside of the x-*coordinates equal to 0. Similarly,* $d_y^s(\gamma\tau)\big|_{\substack{x=0 \\ y=0}}$ *is of the form*

$$d_x^s(\gamma\tau)|_{\substack{x=0 \\ y=0}} = \begin{bmatrix} 0 & 0 \\ * & 0 \end{bmatrix},$$

where $*$ *is the block of* y-*coordinates.*

Proof. Let us argue for the case of x-coordinates only. The case $s = 1$ follows directly from Lemma 4.1, since the identity

$$(14) \qquad\qquad d(\gamma\tau) = {}^t\lambda_\gamma(\tau)^{-1} d\tau \mu_\gamma(\tau)^{-1}$$

yields, after specializing to $\gamma = \eta(\gamma_1, \gamma_2)$, setting $x = y = 0$ and projecting onto the dx-coordinates, the identity

$$d_x(\gamma\tau)|_{\substack{x=0 \\ y=0}} = \begin{bmatrix} 0 & {}^t\lambda_{\gamma_1}(\tau_1)^{-1} dx\, \mu_{\gamma_2}(\tau_2)^{-1} \\ 0 & 0 \end{bmatrix}.$$

The case of $s > 1$ is similar, only starting with an identity obtained by differentiating Equation (14) multiple times. $\qquad\square$

Proposition 4.3. *In the situation of Theorem 2.14, if $\mathrm{d}_x^s f|_{\mathcal{H}^{(1)} \times \mathcal{H}^{(2)}}$ vanishes for all $s < r$ then $\mathrm{d}_{x,\lambda}^r f|_{\mathcal{H}^{(1)} \times \mathcal{H}^{(2)}}$ satisfies the modularity rule*

$$\mathrm{d}_{x,\lambda}^r f|_{\mathcal{H}^{(1)} \times \mathcal{H}^{(2)}}(\tau_1, \tau_2) =$$
$$= \left(\rho_{1,(k,l)}^{+,\lambda}(\lambda_{\gamma_1}(\tau_1), \mu_{\gamma_1}(\tau_1)) \otimes \rho_{2,(k,l)}^{-,\lambda}(\lambda_{\gamma_2}(\tau_2), \mu_{\gamma_2}(\tau_2)) \right)^{-1} \mathrm{d}_{x,\lambda}^r f|_{\mathcal{H}^{(1)} \times \mathcal{H}^{(2)}}(\gamma_1 \tau_1, \gamma_2 \tau_2)$$

where $\tau_1 \in \mathcal{H}^{(1)}, \tau_2 \in \mathcal{H}^{(2)}, \gamma_1 \in \Gamma_1$ and $\gamma_2 \in \Gamma_2$.

Similarly, assuming $\mathrm{d}_y^s f|_{\mathcal{H}^{(1)} \times \mathcal{H}^{(2)}}$ vanishes for all $s < r$, the form $\mathrm{d}_{y,\lambda}^r f|_{\mathcal{H}^{(1)} \times \mathcal{H}^{(2)}}$ satisfies the analogous modularity rule with $\rho_{1,(k,l)}^{+,\lambda}$ replaced by $\rho_{1,(k,l)}^{-,\lambda}$ and with $\rho_{2,(k,l)}^{-,\lambda}$ replaced by $\rho_{2,(k,l)}^{+,\lambda}$.

Proof. Let us argue for the operator $\mathrm{d}_x^r f|_{\substack{x=0 \\ y=0}}$ only (the proof for $\mathrm{d}_y^r f\big|_{\substack{x=0 \\ y=0}}$ is completely analogous). Fix the element $\gamma = \eta(\gamma_1, \gamma_2) \in \Gamma$ and related notation just as in Remark 2.13, and note that for $\tau = \iota(\tau_1, \tau_2)$, we have $\gamma\tau = \iota(\gamma_1 \tau_1, \gamma_2 \tau_2)$ and

$$\lambda_\gamma(\tau) = \begin{bmatrix} \lambda_{\gamma_1}(\tau_1) & \\ & \lambda_{\gamma_2}(\tau_2) \end{bmatrix}, \quad \mu_\gamma(\tau) = \begin{bmatrix} \mu_{\gamma_1}(\tau_1) & \\ & \mu_{\gamma_2}(\tau_2) \end{bmatrix}.$$

Let us rewrite the modular identity

$$(15) \qquad\qquad f(\tau) = \det(\lambda_\gamma(\tau))^{-k} \det(\mu_\gamma(\tau))^{-l} f(\gamma\tau)$$

as

$$\det(\lambda_\gamma(\tau))^k \det(\mu_\gamma(\tau))^l f(\tau) = f(\gamma\tau)$$

Applying the operator $\mathrm{d}_x^r(-)|_{\substack{x=0 \\ y=0}}$ then yields

$$\det(\lambda_\gamma(\tau))^k \det(\mu_\gamma(\tau))^l \mathrm{d}_x^r f\left(\begin{bmatrix} \tau_1 & \\ & \tau_2 \end{bmatrix} \right) = \mathrm{d}_x^r \left(f(\gamma\tau) \right)|_{\substack{x=0 \\ y=0}},$$

since all the remaining terms on the left-hand side coming from the product rule contain $\mathrm{d}_x^s f\left(\begin{bmatrix} \tau_1 & \\ & \tau_2 \end{bmatrix} \right)$ for some $s < r$ and hence vanish. Rearranging the resulting equation then yields

$$\mathrm{d}_x^r f\left(\begin{bmatrix} \tau_1 & \\ & \tau_2 \end{bmatrix} \right) = \det(\lambda_\gamma(\tau))^{-k} \det(\mu_\gamma(\tau))^{-l} \mathrm{d}_x^r \left(f(\gamma\tau) \right)|_{\substack{x=0 \\ y=0}}$$
$$= \det(\lambda_{\gamma_1}(\tau_1))^{-k} \det(\lambda_{\gamma_2}(\tau_2))^{-k} \det(\mu_{\gamma_1}(\tau_1))^{-l} \det(\mu_{\gamma_2}(\tau_2))^{-l} \mathrm{d}_x^r \left(f(\gamma\tau) \right)|_{\substack{x=0 \\ y=0}}.$$

By the chain rule for $\mathrm{d}^r\left(f(\gamma\tau) \right)$, we have

$$\mathrm{d}^r \left(f(\gamma\tau) \right) = \sum_{a=1}^{r} \sum_{b_1+b_2+\cdots+b_a=r} \mathrm{d}^a f \circ \left(\mathrm{d}^{b_1}(\gamma\tau), \mathrm{d}^{b_2}(\gamma\tau), \ldots, \mathrm{d}^{b_a}(\gamma\tau) \right),$$

which gives

$$
\begin{aligned}
\mathrm{d}_x^r \left(f(\gamma\tau) \right)\big|_{\substack{x=0 \\ y=0}} &= \sum_{a=1}^{r} \sum_{b_1+b_2+\cdots+b_a=r} \left(\mathrm{d}^a f\big|_{\substack{x=0 \\ y=0}} \right) \circ \left(\mathrm{d}_x^{b_1}(\gamma\tau), \mathrm{d}_x^{b_2}(\gamma\tau), \ldots, \mathrm{d}_x^{b_a}(\gamma\tau) \right)\Big|_{\substack{x=0 \\ y=0}} \\
&= \sum_{a=1}^{r} \sum_{b_1+b_2+\cdots+b_a=r} \left(\mathrm{d}_x^a f\big|_{\substack{x=0 \\ y=0}} \right) \circ \left(\mathrm{d}_x^{b_1}(\gamma\tau), \mathrm{d}_x^{b_2}(\gamma\tau), \ldots, \mathrm{d}_x^{b_a}(\gamma\tau) \right)\Big|_{\substack{x=0 \\ y=0}} \\
&= \left(\mathrm{d}_x^r f\big|_{\substack{x=0 \\ y=0}} \right) \circ \left(\mathrm{d}_x(\gamma\tau), \mathrm{d}_x(\gamma\tau), \ldots, \mathrm{d}_x(\gamma\tau) \right)\big|_{\substack{x=0 \\ y=0}},
\end{aligned}
$$

where the second equality follows from Corollary 4.2 and the third one from the assumption that $\mathrm{d}_x^a f\big|_{\substack{x=0 \\ y=0}} = 0$ when $a < r$. Lemma 4.1 now leads to the expression

$$
\begin{aligned}
&\mathrm{d}_x^r \left(f(\gamma\tau) \right)\big|_{\substack{x=0 \\ y=0}} \\
&= (\mathrm{d}_x^r f)\left(\begin{bmatrix} \gamma_1\tau_1 & \\ & \gamma_2\tau_2 \end{bmatrix} \right) \circ \left({}^t\lambda_{\gamma_1}(\tau_1)^{-1}\mathrm{d}x\, \mu_{\gamma_2}(\tau_2)^{-1}, \ldots, {}^t\lambda_{\gamma_1}(\tau_1)^{-1}\mathrm{d}x\, \mu_{\gamma_2}(\tau_2)^{-1} \right).
\end{aligned}
$$

Then, by definition of ρ' as in (6), we further have

$$
\mathrm{d}_x^r \left(f(\gamma\tau) \right)\big|_{\substack{x=0 \\ y=0}} = \rho'(\lambda_{\gamma_1}(\tau_1)^{-1}, \mu_{\gamma_2}(\tau_2)^{-1}) \left(\mathrm{d}_x^r f\left(\begin{bmatrix} \gamma_1\tau_1 & \\ & \gamma_2\tau_2 \end{bmatrix} \right) \right),
$$

and altogether, we obtain

$$
\begin{aligned}
\mathrm{d}_x^r f\left(\begin{bmatrix} \tau_1 & \\ & \tau_2 \end{bmatrix} \right) = {}&\det(\lambda_{\gamma_1}(\tau_1))^{-k} \det(\lambda_{\gamma_2}(\tau_2))^{-k} \det(\mu_{\gamma_1}(\tau_1))^{-l} \det(\mu_{\gamma_2}(\tau_2))^{-l} \\
&\times \rho'(\lambda_{\gamma_1}(\tau_1)^{-1}, \mu_{\gamma_2}(\tau_2)^{-1})\mathrm{d}_x^r f\left(\begin{bmatrix} \gamma_1\tau_1 & \\ & \gamma_2\tau_2 \end{bmatrix} \right).
\end{aligned}
$$

Finally, projecting onto the $\mathbb{S}^\lambda \boxtimes \mathbb{S}^\lambda$-component of ρ' yields the desired result. $\qquad\square$

4.2. **Holomorphicity at cusps and tensor product decomposition.** Proposition 4.3 shows that $\mathrm{d}_{x,\lambda}^r f|_{\mathcal{H}^{(1)} \times \mathcal{H}^{(2)}}$ yields a vector-valued function that transforms the same way as the tensor product of automorphic forms in Theorem 2.14. To conclude that $\mathrm{d}_{x,\lambda}^r f|_{\mathcal{H}^{(1)} \times \mathcal{H}^{(2)}}$ *is* such a tensor product, we employ the following linear-algebraic lemma, going back to Witt [Wit41].

Lemma 4.4. *Consider a map $F : X \times Y \to V \otimes_{\mathbb{C}} W$ where V, W are finite-dimensional $\mathbb{C}$-vector spaces and X, Y are arbitrary sets. Let L_X (L_Y, resp.) be a chosen finite-dimensional subspace of maps $X \to V$ ($Y \to W$, resp.). Fix a choice of basis $\{b_i\}_{i=1}^n$ of V and $\{c_j\}_{j=1}^m$ of W, and assume that*

(1) for all $y \in Y$ and all j, the projection of $F|_{X \times \{y\}}$ onto $V \otimes c_j \simeq V$ belongs to L_X,
(2) for all $x \in X$ and all i, the projection of $F|_{\{x\} \times Y}$ onto $b_i \otimes W \simeq W$ belongs to L_Y.
Then F can be written in the form

$$
F = \sum_k G_k \otimes H_k, \quad G_k \in L_X, \; H_k \in L_Y.
$$

Before proceeding with the proof, we note that the assumptions of Lemma 4.4 are independent of the choices of bases.

Proof. When V and W are one-dimensional, we may identify V, W and $V \otimes W$ with $\mathbb{C}$. Then the claim is the content of [Wit41, Satz A]. In general, expressing all the involved vector functions as coordinate functions with respect to the bases $\{b_i\}_{i=1}^n$ of V, $\{c_j\}_{j=1}^m$ of W and $\{b_i \otimes c_j\}_{i,j}$ of $V \otimes W$, resp.,the vector-valued functions $X \to V$ ($Y \to W$ and $X \times Y \to V \otimes W$, resp.) can be treated as scalar-valued functions $X \times \{1, \ldots, n\} \to \mathbb{C}$ ($Y \times \{1, \ldots, m\} \to \mathbb{C}$ and $X \times Y \times \{1, \ldots, n\} \times \{1, \ldots, m\} \to \mathbb{C}$, resp.) in the obvious manner. This reduces the claim of the Lemma to the scalar-valued case. $\qquad\square$

Proposition 4.5. *In the situation of Proposition 4.3, the form $\mathrm{d}^r_{x,\lambda} f|_{\mathcal{H}^{(1)} \times \mathcal{H}^{(1)}}$ satisfies the assumptions of Lemma 4.4 with*

$$X = \mathcal{H}^{(1)}, \quad L_X = M_{\rho^{+,\lambda}_{1,(k,l)}}(\Gamma_1) \quad and \quad Y = \mathcal{H}^{(2)}, \quad L_Y = M_{\rho^{-,\lambda}_{2,(k,l)}}(\Gamma_2).$$

Similarly, the form $\mathrm{d}^r_{y,\lambda} f|_{\mathcal{H}^{(1)} \times \mathcal{H}^{(1)}}$ satisfies the assumptions of Lemma 4.4 with

$$X = \mathcal{H}^{(1)}, \quad L_X = M_{\rho^{-,\lambda}_{1,(k,l)}}(\Gamma_1) \quad and \quad Y = \mathcal{H}^{(2)}, \quad L_Y = M_{\rho^{+,\lambda}_{2,(k,l)}}(\Gamma_2).$$

Proof. As long as neither of the unitary groups $\mathrm{U}_1, \mathrm{U}_2$ is of signature $(1, 1)$ or quasi–split over $\mathbb{Q}$, to verify whether the form $\mathrm{d}^r_{x,\lambda} f|_{\mathcal{H}^{(1)} \times \mathcal{H}^{(1)}}$ or $\mathrm{d}^r_{y,\lambda} f|_{\mathcal{H}^{(1)} \times \mathcal{H}^{(1)}}$, after restriction and projection as in Lemma 4.4, produces automorphic forms of the indicated level and weight comes down to verifying the appropriate modularity rule. In this case, the conclusion immediately follows from Proposition 4.3.

When U_1 or U_2 is of signature $(1, 1)$ and is quasi-split over $\mathbb{Q}$, we additionally need to verify the holomorphicity at cusps condition. Note that in this case, there is no need to take any projection to $\mathbb{S}^\lambda$ components, and we therefore suppress λ from the notation to simplify from now on (cf. Remark 3.3).

Let us assume that U_1 is of signature $(1, 1)$ and is quasi-split over $\mathbb{Q}$, fix $\tau_2 \in \mathcal{H}^{(2)}$ and let us verify the holomorphicity at cusps in the case of $\mathrm{d}^r_x f|_{\mathcal{H}^{(1)} \times \{\tau_2\}}$ and $\mathrm{d}^r_y f|_{\mathcal{H}^{(1)} \times \{\tau_2\}}$. The arguments are the same in the remaining cases. Acting on f by $\eta(\beta)$ where $\beta \in \mathrm{SL}_2(\mathbb{Q})$, it is enough to verify holomorphicity at ∞.

We consider first the case (UT), i.e. the situation when $\mathrm{U}(\mathbb{R})$ is identified with $\mathrm{U}(\eta_n)$ and U is itself quasi-split. In this case, it is enough to even consider $\mathrm{d}^r_x f|_{\mathcal{H}^{(1)} \times \{\tau_2\}}$ only, as the reasoning for $\mathrm{d}^r_y f|_{\mathcal{H}^{(1)} \times \{\tau_2\}}$ is completely symmetrical. We consider the Fourier expansion of f written as follows,

$$(16) \qquad f(\tau) = \sum_h c(h) \exp(2\pi i(h_1 \tau_1 + \mathrm{tr}\,{}^t\overline{h_3} y + \mathrm{tr}\, h_3 x + \mathrm{tr}\, h_2 \tau_2)),$$

where $h = \begin{bmatrix} h_1 & {}^t\overline{h_3} \\ h_3 & h_2 \end{bmatrix}$ ranges over the appropriate lattice of Hermitian matrices, with h_1 a number and h_2 a block of size $(n-1, n-1)$. Then we have

$$\mathrm{d}^r_x f(\tau) = (2\pi i)^r \sum_h \sum_\alpha c(h) h_3^\alpha \exp(2\pi i(\mathrm{tr}\, h_1 \tau_1 + \mathrm{tr}\,{}^t\overline{h_3} y + \mathrm{tr}\, h_3 x + \mathrm{tr}\, h_2 \tau_2)) \mathrm{d}x_\alpha,$$

where $\alpha = (i_1, i_2, \ldots, i_r)$ is a multi-index and h_3^α denotes $h_3^{(i_1)} h_3^{(i_2)} \ldots h_3^{(i_r)}$, the product of respective entries of the row vector h_3.

Consequently, we have

(17)

$$\mathrm{d}_x^r f|_{\substack{x=0 \\ y=0}} = (2\pi i)^r \sum_h \sum_\alpha c(h) h_3^\alpha \mathrm{d}x_\alpha \exp(2\pi i(h_1\tau_1 + \mathrm{tr}\, h_2\tau_2))$$

$$(18) \qquad = \sum_\alpha \sum_{h_1} \underbrace{\left((2\pi i)^r \sum_{h_2,h_3} c\left(\begin{bmatrix} h_1 & {}^t\overline{h_3} \\ h_3 & h_2 \end{bmatrix}\right) h_3^\alpha \exp(2\pi i\mathrm{tr}(h_2\tau_2)) \right)}_{C(h_1,\alpha)} \exp(2\pi i h_1\tau_1)\mathrm{d}x_\alpha,$$

where for fixed τ_2 and α, the terms $C(h_1,\alpha)$ are the Fourier coefficients for $\mathrm{d}_x^r f|_{\mathcal{H}^{(i)} \times \{\tau_2\}}$ projected onto $\mathrm{d}x_\alpha$. It follows that such a coefficient indexed by h_1 can be nonzero only if h_1 fits into a positive-semidefinite Hermitian matrix $\begin{bmatrix} h_1 & {}^t\overline{h_3} \\ h_3 & h_2 \end{bmatrix}$. In particular, in this case $h_1 \geq 0$, which proves the claim.

In the case (UB), we proceed similarly using Fourier–Jacobi expansions. Let us assume $m > n$, and utilize a change of coordinates on U according to Section 2.3. That is, we may treat f as a function $f(\tau,u)$ on the symmetric space $\widetilde{\mathcal{H}}_{m,n}$ instead, and consider the variant of the construction outlined in Remark 2.16. In the notation introduced therein, the Fourier–Jacobi expansion takes the following form:

$$(19) \qquad f(\tau,u) = \sum_h c(w,u_2;h) \exp(2\pi i(h_1\tau_1 + \mathrm{tr}\, {}^t\overline{h_3}z + \mathrm{tr}\, h_3 x + \mathrm{tr}\, h_2\tau_2))$$

(recall from Remark 2.16 that z, w are names for y-coordinates based on whether they come from τ or u). In the case of the operator $\mathrm{d}_x^r(-)$, the argument above applies almost verbatim, replacing $c(h)$ with $c(w,u_2;h)$, $\mathrm{tr}\, {}^t\overline{h_3}y$ with $\mathrm{tr}\, {}^t\overline{h_3}z$, etc.

In the case of the operator $\mathrm{d}_y^r(-)$, the same argument still applies, but the formula for the resulting coefficients $C(h_1, \bullet)$ is more involved; namely, we have

$$C(h_1,\alpha,\beta) = \sum_{h_2,h_3} \partial_{w_\beta} c\left(0, u_2; \begin{bmatrix} h_1 & h_3 \\ {}^t\overline{h_3} & h_2 \end{bmatrix} \right) (2\pi i)^{|\alpha|} h_3^\alpha \exp(2\pi i\mathrm{tr}(h_2\tau_2))\mathrm{d}z_\alpha \mathrm{d}w_\beta.$$

Here α, β are again multi-indices with $|\alpha|+|\beta| = r$, where $|\alpha|, |\beta|$ denotes their lengths. The key point is that when the matrix h is not positive-semidefinite, the coefficient functions $c(u;h)$ are identically zero functions of u, and therefore so are all the partial derivatives $\partial_{w_\beta} c(-;h)$ appearing in the formula.

Finally, the remaining case is when U is of equal signature (n,n), but not itself quasi-split. The argument in this case uses the second variant of coordinates listed in Section 2.3, but otherwise goes along the same lines as the above two variants. To avoid excessive repetition, we leave this case to the reader. $\qquad\square$

Proof of Theorem 2.14. Theorem 2.14 follows directly as a combination of Lemma 4.4 and Proposition 4.5. Let us only stress the point that the spaces L_X, L_Y taken in Proposition 4.5 are finite-dimensional, so that Lemma 4.4 applies. $\qquad\square$

We finish this section with the observation that our construction produces cusp forms out of cusp forms.

Proposition 4.6. *In the situation of Theorem 2.14, assume that we are in the case (UT) and that f is a cusp form. Then the decomposition of $\mathrm{d}^r_{x,\lambda}f\big|_{\mathcal{H}^{(1)}\times\mathcal{H}^{(2)}}$ can be written in the form $\sum f_k \otimes F_k$, where all the forms f_k, F_k are cusp forms of appropriate levels and weights. Similarly, in the decomposition $\mathrm{d}^r_{y,\lambda}f\big|_{\mathcal{H}^{(1)}\times\mathcal{H}^{(2)}} = \sum g_k \otimes G_k$, all the forms g_k, G_k can be taken as cusp forms.*

Proof. We may repeat the proofs of Proposition 4.5 and Theorem 2.14 almost verbatim, with the following two adjustments:

(1) In the Fourier expansion for f (Equation (16)), one has $c(h) \neq 0$ only when the hermitian matrix h is positive-definite (rather than non-negative). As a result, writing again

$$h = \begin{bmatrix} h_1 & {}^t\overline{h_3} \\ h_3 & h_1 \end{bmatrix}$$

for h_1, h_2 hermitian matrices of the appropriate dimensions, the coefficients in the analogue of Equation (18) are nonzero only when h_1, h_2 are positive-definite.

(2) As a result, we conclude an analogue of Proposition 4.5 (hence an analogue of proof of Theorem 2.14) with the choice of L_X and L_Y as the spaces of cusp forms (of the indicated level and weight) instead of the full spaces of automorphic forms.

$\square$

5. Algebraic modular forms on unitary groups.

In this section, we review the notion of modular forms for unitary groups as global section of certain automorphic vector bundles over Shimura varieties. We follow the approach of [EHLS20, Section 2] and [EFMV18, Section 2], only recalling the notions relevant to the current article. The aim is to rephrase the differential operators above in an algebraic context.

5.1. **Shimura varieties associated to unitary groups.** Let V be a Hermitian K-vector space, and let $G = GU(V)$ be the associated similitude unitary group, as in Section 2.1.

Fix a complex embedding $\sigma : K \hookrightarrow \mathbb{C}$, let $V_{\mathbb{C}} = V_\sigma := V \otimes_{K,\sigma} \mathbb{C}$. We identify V_σ with $V \otimes_{\mathbb{Q}} \mathbb{R}$. The signature (m, n) introduced in Section 2.1 is simply the signature of the Hermitian pairing $\langle \cdot, \cdot \rangle_{\mathbb{C}}$ on $V_{\mathbb{C}}$. By choosing the conjugate embedding $\overline{\sigma} = \sigma \circ c$, one obtains the signature (n, m) instead.

Fix a basis of $V_{\mathbb{C}} = V \otimes \mathbb{R}$ such that the quadratic form associated to $\langle \cdot, \cdot \rangle_{\mathbb{C}}$ is $Q_V(x_1, \ldots, x_d) = x_1^2 + \ldots x_m^2 - x_{m+1}^2 - \ldots - x_d^2$. With respect to this basis, define $h : \mathbb{C} \to \mathrm{End}_{K \otimes \mathbb{R}}(V \otimes \mathbb{R})$ via

$$z \mapsto \mathrm{diag}(z1_m, \overline{z}1_n).$$

Let X denote the set of all $G(\mathbb{R})$-conjugate of h induced by the natural action of $G(\mathbb{R})$ on $\mathrm{End}_{\mathbb{R}}(V_{\mathbb{C}}) = \mathrm{End}_{\mathbb{R}}(V \otimes \mathbb{R})$.

The space X is the locally symmetric space associated to G and it is well known that the pair (G, X) defines a *Shimura datum* in the usual sense. Let $E = K$ if $m \neq n$ and $E = \mathbb{Q}$ if $m = n$, i.e. E is the *reflex field* of (G, X).

Given a *neat* compact open subgroup $\mathcal{U} \subset G(\mathbb{A}_f)$, the double coset space

$$\mathrm{Sh}_{\mathcal{U}}(G, X)(\mathbb{C}) := G(\mathbb{Q}) \backslash X \times G(\mathbb{A}_f) / \mathcal{U}$$

corresponds to the $\mathbb{C}$-points of a smooth quasi-projective scheme $\mathrm{Sh}_{\mathcal{U}}(G, X)$ over E. Here, we do not specify the exact definition of *neat* subgroups, although it can be understood as "sufficiently small".

We refer to $\mathrm{Sh}_{\mathcal{U}}(G, X)$ as the *Shimura variety* associated to (G, X) of level $\mathcal{U}$ and denote it $\mathrm{Sh}_{\mathcal{U}}$ when the datum (G, X) is clear from context.

For our purpose, note that $\mathrm{Sh}_{\mathcal{U}}$ is a certain (canonical) choice of a connected component of a moduli space $\mathrm{M}_{\mathcal{U}} = \mathrm{M}_{\mathcal{U}}(G, X)$ over E representing a moduli problem associated to G and h. The moduli problem classifies abelian varieties A with *extra structures* (depending on G, h and $\mathcal{U}$), including an embedding $i : K \hookrightarrow \mathrm{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q}$, see [EFMV18, Section 2.2].

Most importantly, there is a canonical inclusion $s_{\mathcal{U}} : \mathrm{Sh}_{\mathcal{U}} \hookrightarrow \mathrm{M}_{\mathcal{U}}$ and a universal abelian variety $A_{\mathcal{U}}^{univ} \to \mathrm{M}_{\mathcal{U}}$ (with extra structure).

Let $\underline{\Omega}_{\mathcal{U}}$ denote the sheaf of relative differentials of $A_{\mathcal{U}}^{univ}$ over $\overline{\mathrm{M}}_{\mathcal{U}}$ and define $\underline{\omega}_{\mathcal{U}} := \pi_* \underline{\Omega}_{\mathcal{U}}$. It is well known that $\underline{\omega}_{\mathcal{U}}$ is locally free of rank $\dim V = m + n$, and it admits a natural action by $K \otimes_{\mathbb{Q}} K$ induced by the embedding $i_{\mathcal{U}}^{univ}$ associated to $A_{\mathcal{U}}^{univ}$ and the structure of $\mathrm{M}_{\mathcal{U}}$ as a scheme over K.

5.2. Algebraic modular forms.
By identifying $K \otimes K \cong \sigma(K) \oplus \overline{\sigma}(K)$ as a subset of $\mathbb{C} \oplus \mathbb{C}$, one obtains a decomposition

$$\underline{\omega}_{\mathcal{U}} = \underline{\omega}_{\mathcal{U}}^+ \oplus \underline{\omega}_{\mathcal{U}}^-,$$

where $z \in K$ acts on $\underline{\omega}_{\mathcal{U}}^+$ (resp. $\underline{\omega}_{\mathcal{U}}^-$) via $\sigma(z)$ (resp. $\overline{\sigma}(z)$) and $\underline{\omega}_{\mathcal{U}}^+$ (resp. $\underline{\omega}_{\mathcal{U}}^-$) has rank m (resp. n).

Let $\mathcal{E}_{\mathcal{U}}$ denote the sheaf

$$\mathrm{Isom}_{\mathcal{O}_{\mathrm{M}_{\mathcal{U}}}}((\mathcal{O}_{\mathrm{M}_{\mathcal{U}}})^m, \underline{\omega}_{\mathcal{U}}^+) \oplus \mathrm{Isom}_{\mathcal{O}_{\mathrm{M}_{\mathcal{U}}}}((\mathcal{O}_{\mathrm{M}_{\mathcal{U}}})^n, \underline{\omega}_{\mathcal{U}}^-)$$

over $\mathrm{M}_{\mathcal{U}}$. In particular, there is a natural (left) action by the algebraic group $H = \mathrm{GL}(m) \times \mathrm{GL}(n)$ on $\mathcal{E}_{\mathcal{U}}$.

Given an algebraic representation (ρ, W) of H over E, let $\mathcal{E}_{\mathcal{U}}^{\rho}$ denote the sheaf whose R-points are given by

$$\mathcal{E}_{\mathcal{U}}^{\rho}(R) = (\mathcal{E}_{\mathcal{U}}(R) \times (W \otimes_E R)) / ((\varepsilon, v) \sim (l\varepsilon, \rho({}^t l^{-1}) v)),$$

for all maps $\mathrm{Spec}\, R \to \mathcal{E}_{\mathcal{U}}$.

Example 5.1. An element of the space $\mathcal{E}_{\mathcal{U}}^+ := \mathrm{Isom}_{\mathcal{O}_{\mathrm{M}_{\mathcal{U}}}}((\mathcal{O}_{\mathrm{M}_{\mathcal{U}}})^m, \underline{\omega}_{\mathcal{U}}^+)$ provides a (local) basis of $\underline{\omega}_{\mathcal{U}}^+$ and therefore identifies the fibres of $\underline{\omega}_{\mathcal{U}}^+$ with the algebraic representation $\rho_{\mathrm{std}}^+ = \rho_{\mathrm{std}} \otimes \mathbf{1}$ of H.

In other words, $\mathcal{E}_{\mathcal{U}}^+$ is canonically isomorphic to the sheaf $\mathcal{E}_{\mathcal{U}}^{\rho_{\mathrm{std}}^+}$ over $\mathrm{M}_{\mathcal{U}}$ associated to ρ_{std}^+. Similarly, $\mathcal{E}_{\mathcal{U}}^- := \mathrm{Isom}_{\mathcal{O}_{\mathrm{M}_{\mathcal{U}}}}((\mathcal{O}_{\mathrm{M}_K})^n, \underline{\omega}_{\mathcal{U}}^-)$ is naturally identified with $\mathcal{E}_{\mathcal{U}}^{\rho_{\mathrm{std}}^-}$, where $\rho_{\mathrm{std}}^- := \mathbf{1}_{\mathrm{GL}(m)} \otimes \rho_{\mathrm{std},\mathrm{GL}(n)}$.

Now, consider the maximal torus T of H consisting of diagonal matrices. We identify the group of algebraic characters of T with $\mathbb{Z}^{\oplus(n+m)}$ and denote an algebraic κ of T by a tuple of integers $\kappa = (\kappa_1, \ldots, \kappa_m; \kappa_{m+1}, \ldots, \kappa_{m+n})$. The identification is given by

$$\kappa : T \to \mathbb{G}_{\mathrm{m}}$$

$$Z \mapsto \prod_{i=1}^{d} z_i^{\kappa_i}\,,$$

where $Z = (\mathrm{diag}(z_1, \ldots, z_m), \mathrm{diag}(z_{m+1}, \ldots, z_{m+n}))$.

One says that κ is dominant (with respect to the standard upper triangular Borel subgroup of H) if

$$\kappa_1 \geq \ldots \geq \kappa_m \quad \text{and} \quad \kappa_{m+1} \geq \ldots \geq \kappa_{m+n}\,.$$

For each dominant weight κ of T, there is a unique (up to isomorphism) algebraic representation $(\rho, W) = (\rho_\kappa, W_\kappa)$ of H over K whose highest weight is κ, see [EHLS20, Section 2.6.3] or [Jan03].

Definition 5.2. The space of modular forms over G of level $\mathcal{U}$ and weight κ is defined as the space of global sections

$$M_\kappa(\mathcal{U}; R) := H^0(\mathrm{Sh}_{\mathcal{U}/R}, s_{\mathcal{U}}^* \mathcal{E}_{\mathcal{U}}^{\rho_\kappa})\,,$$

for any E-algebra R, where $s_{\mathcal{U}} : \overline{\mathrm{Sh}}_{\mathcal{U}} \to \overline{\mathrm{M}}_{\mathcal{U}}$ is as in the previous section.

Example 5.3. The algebraic representation $\rho_{\mathrm{std}}^+ = \rho_{\mathrm{std},\mathrm{GL}(m)} \otimes \mathbf{1}_{\mathrm{GL}(n)}$ of H from Example 5.1 is the representation of highest weight $\kappa_{\mathrm{std}}^+ := (1, 0, \ldots, 0; 0, \ldots, 0)$. Similarly, $\rho_{\mathrm{std}}^- = \mathbf{1}_{\mathrm{GL}(m)} \otimes \rho_{\mathrm{std},\mathrm{GL}(n)}$ has highest weight $\kappa_{std}^- := (0, \ldots, 0; 1, 0, \ldots, 0)$.

Example 5.4. Given two integers k, l, let $\kappa_{k,l} = (k, \ldots, k; l, \ldots, l)$ with m-many k's and n-many l's. If the signature of the underlying group G is clear from context, we simply write $\kappa_{k,l} = (k, l)$.

The corresponding highest weight representation is

$$\Delta_{k,l} := \rho_{\kappa_{k,l}} = \left(\bigwedge^m \rho_{\mathrm{std}}^+ \right)^{\otimes k} \boxtimes \left(\bigwedge^n \rho_{\mathrm{std}}^- \right)^{\otimes l} \cong \det_{\mathrm{GL}(m)}^{\otimes k} \boxtimes \det_{\mathrm{GL}(n)}^{\otimes l}\,,$$

as in Example 2.4. We write $\mathcal{E}_{\mathcal{U}}^{(k,l)}$ for the corresponding sheaf $\mathcal{E}_{\mathcal{U}}^{\Delta_{k,l}}$. From [EM21, Section 2.5.1], we see that $\mathcal{E}_{\mathcal{U}}^{(k,l)}$ isomorphic to

$$\underline{\omega}_{\mathcal{U}}^{(k,l)} := \left(\bigwedge^m \underline{\omega}_{\mathcal{U}}^+ \right)^{\otimes k} \otimes \left(\bigwedge^n \underline{\omega}_{\mathcal{U}}^- \right)^{\otimes l}\,.$$

5.3. **Schur functor.** Let ρ_{std}^+ and ρ_{std}^- denote the m-dimensional and resp. n-dimensional standard representation of $\mathrm{GL}(m)$ and $\mathrm{GL}(n)$, respectively.

We implicitly fix the standard basis of each of these representations to identify ρ_{std}^+ with $\mathbb{C}^{\oplus m}$ and ρ_{std}^- with $\mathbb{C}^{\oplus n}$. To construct and study the differential operators in this paper, we could fix any other basis. This canonical choice is simply convenient.

The above provides a basis for $\rho_{\mathrm{std}} := \rho_{\mathrm{std}}^+ \boxtimes \rho_{\mathrm{std}}^-$ and more generally, a basis for $W_\kappa = \mathbb{S}_\kappa(\rho_{\mathrm{std}})$, for each dominant weight κ of H. Here, $\mathbb{S}_\kappa$ is the κ-Schur functor as in [FH13, Section 15.3].

As explained in [EM21, Section 2.2.2.], the functor $\mathbb{S}_\kappa$ on vector spaces naturally extends to a functor on locally free sheaves, again denoted $\mathbb{S}_\kappa$. More precisely, given a locally free sheaf $\mathbb{F}$ of modules over $\mathrm{M}_\mathcal{U}$, we have $\mathbb{S}_\kappa(\mathcal{F})(\mathrm{Spec}\,R) = \mathbb{S}_\kappa(\mathcal{F}(\mathrm{Spec}\,R))$, for any affine open $\mathrm{Spec}\,R$ of $\mathrm{M}_\mathcal{U}$.

As explained in [EM21, Section 2.5.1], the choices of basis above induce an isomorphism between $\underline{\omega}^\kappa := \mathbb{S}_\kappa(\underline{\omega})$ and $\mathcal{E}^{\rho_\kappa}$. Note that Example 5.4 is a particular case of this identification.

Therefore, in what follows, we pass between the notation $\mathcal{E}^\bullet$ and $\underline{\omega}_\bullet$ on Shimura varieties for all dominant weights, without comments. This is mainly used for weights corresponding to symmetric powers of standard representations.

5.4. **Gauss–Manin connection.** We now introduce the Gauss–Manin connection in order to give a definition of differential operators. These definitions agree with those in the previous section, when we look at the effect on local coordinates given by the covering space $\mathcal{H}_{m,n}$.

Let S be a smooth scheme over K, and $\pi : X \to S$ be a smooth proper morphism of schemes. Let $\mathcal{H}_{dR}^q(X/S)$ be the cohomology sheaves of the (relative) de Rham complex of X/S. As explained in [EFMV18, Section 3.1], the *Gauss–Manin connection* ∇ is an integrable connection

$$\nabla : \mathcal{H}_{dR}^q(X/S) \to \mathcal{H}_{dR}^q(X/S) \otimes_{\mathcal{O}_S} \Omega^1(S).$$

If we specialize this to $q = 1$, $S = \mathrm{M}_\mathcal{U}$, and $X = A_\mathcal{U}^{univ}$, we obtain

$$\nabla : \mathcal{H}_{dR}^1(A_\mathcal{U}^{univ}/\mathrm{M}_\mathcal{U}) \to \mathcal{H}_{dR}^1(A_\mathcal{U}^{univ}/\mathrm{M}_\mathcal{U}) \otimes \Omega_{\mathrm{M}_\mathcal{U}/K}.$$

We now want to understand how this interacts with the sheaves $\underline{\omega}_\mathcal{U}$. Our first observation is that $\underline{\omega}_\mathcal{U}$ is a subsheaf of $\mathcal{H}_{dR}^1(A_\mathcal{U}^{univ}/\mathrm{M}_\mathcal{U})$. In fact, we have an exact sequence

$$0 \to \underline{\omega}_\mathcal{U} \to \mathcal{H}_{dR}^1(A_\mathcal{U}^{univ}/\mathrm{M}_\mathcal{U}) \to R^1\pi_* \mathcal{O}_{A^{univ}} \to 0,$$

which splits pointwise on $\mathrm{M}_\mathcal{U}$. Moreover from [EFMV18, Section 3.3.1], we see that we have a splitting over $\mathrm{M}_\mathcal{U}$ (but only when viewed as a C^∞-manifold). In particular, we have a projector

$$\varpi(C^\infty) : \mathcal{H}_{dR}^1(C^\infty) \twoheadrightarrow \omega_\mathcal{U}(C^\infty).$$

Thus, we can in fact get a connection

$$\nabla_\omega(C^\infty) : \underline{\omega}_\mathcal{U}(C^\infty) \to \underline{\omega}_\mathcal{U}(C^\infty) \otimes \Omega_{\mathrm{M}_\mathcal{U}(C^\infty)/K},$$

by restricting ∇ to $\underline{\omega}_\mathcal{U}$, and applying $\varpi(C^\infty) \otimes \mathrm{id}$.

A similar idea allows us to also construct the connection on $\underline{\omega}_\mathcal{U}^{(k,l)}$. Indeed, observe that the $\mathcal{O}_K$ action on $A_\mathcal{U}^{univ}$ induces a decomposition

$$\mathcal{H}_{dR}^1(A_\mathcal{U}^{univ}/\mathrm{M}_\mathcal{U}) \simeq \mathcal{H}_{dR}^{1,+}(A_\mathcal{U}^{univ}/\mathrm{M}_\mathcal{U}) \oplus \mathcal{H}_{dR}^{1,-}(A_\mathcal{U}^{univ}/\mathrm{M}_\mathcal{U}),$$

and moreover

$$\nabla(\mathcal{H}_{dR}^{1,\pm}(A_\mathcal{U}^{univ}/\mathrm{M}_\mathcal{U})) \subseteq \mathcal{H}_{dR}^{1,\pm}(A_\mathcal{U}^{univ}/\mathrm{M}_\mathcal{U}) \otimes \Omega_{\mathrm{M}_\mathcal{U}/K}.$$

Now, we see that for any $s \in \mathbb{N}$, we have a map

$$\nabla_{\otimes s} : \mathcal{H}^1_{dR}(A^{univ}_{\mathcal{U}}/\mathrm{M}_{\mathcal{U}})^{\otimes s} \to \mathcal{H}^1_{dR}(A^{univ}_{\mathcal{U}}/\mathrm{M}_{\mathcal{U}})^{\otimes s} \otimes \Omega_{\mathrm{M}_{\mathcal{U}}/K},$$

which on local sections is given by

$$\nabla_{\otimes s}(f_1 \otimes \cdots \otimes f_s) = \sum_{i=1}^{s} \iota_i(f_1 \otimes \cdots \otimes \nabla(f_i) \otimes \cdots \otimes f_k),$$

where ι_i is the isomorphism which is defined on local sections as

$$\mathcal{H}^1_{dR}(A^{univ}_{\mathcal{U}}/\mathrm{M}_{\mathcal{U}})^{\otimes i} \otimes \Omega_{\mathrm{M}_{\mathcal{U}}/K} \otimes \mathcal{H}^1_{dR}(A^{univ}_{\mathcal{U}}/\mathrm{M}_{\mathcal{U}})^{\otimes s-i} \xrightarrow{\sim} \mathcal{H}^1_{dR}(A^{univ}_{\mathcal{U}}/\mathrm{M}_{\mathcal{U}})^{\otimes s} \otimes \Omega_{\mathrm{M}_{\mathcal{U}}/K}$$

via

$$(e_1 \otimes \cdots \otimes e_i) \otimes u \otimes (e_{i+1} \otimes \cdots \otimes e_d) \mapsto (e_1 \otimes \cdots \otimes e_d) \otimes u \,.$$

In fact, viewieng $\rho_{k,l}$ as a subrepresentation of $(\rho_{\mathrm{std}})^{\otimes s}$, by taking Schur functors and projectors, we in fact have a C^∞-differential operator

$$\nabla : \underline{\omega}^{(k,l)}_{\mathcal{U}} \to \underline{\omega}^{(k,l)}_{\mathcal{U}} \otimes \Omega_{\mathrm{M}_{\mathcal{U}}/K} \xrightarrow{\sim} \underline{\omega}^{(k,l)}_{\mathcal{U}} \otimes \underline{\omega}^+_{\mathcal{U}} \otimes \underline{\omega}^-_{\mathcal{U}}$$

where $\underline{\omega}^{(k,l)}_{\mathcal{U}}$ is as in Example 5.4 and the last map is the Kodaira–Spencer isomorphism

$$\underline{\omega}^+_{\mathcal{U}} \otimes \underline{\omega}^-_{\mathcal{U}} \xrightarrow{\sim} \Omega_{\mathrm{M}_{\mathcal{U}}/K} \,,$$

see [Lan18, Proposition 3.4.3.3].

In particular, we have a map on global sections

$$\mathbf{d} : H^0(\mathrm{M}_{\mathcal{U}}(C^\infty), \underline{\omega}^{(k,l)}_{\mathcal{U}}(C^\infty)) \to H^0(\mathrm{M}_{\mathcal{U}}(C^\infty), (\underline{\omega}^{(k,l)}_{\mathcal{U}} \otimes \underline{\omega}^+_{\mathcal{U}} \otimes \underline{\omega}^-_{\mathcal{U}})(C^\infty)) \,.$$

Note that if we start with a holomorphic section $H^0(\mathrm{M}_{\mathcal{U}}, \underline{\omega}^{(k,l)}_{\mathcal{U}})$, then the image under this map is also holomorphic, (if it is vector valued, then we are asking that the function is coordinate-wise holomorphic) and hence lies in $H^0(\mathrm{M}_{\mathcal{U}}, \underline{\omega}^{(k,l)}_{\mathcal{U}} \otimes \underline{\omega}^+_{\mathcal{U}} \otimes \underline{\omega}^-_{\mathcal{U}})$.

One can iterate $\mathbf{d}$ consecutively r times to obtain smooth sections of $\underline{\omega}^{(k,l)}_{\mathcal{U}} \otimes (\underline{\omega}^+_{\mathcal{U}} \otimes \underline{\omega}^-_{\mathcal{U}})^{\otimes r}$. Moreover, we observe that since we are differentiating smooth sections, we have symmetry of mixed partial derivatives, hence the image of $\mathbf{d}^r$ is in fact actually contained in $\underline{\omega}^{(k,l)}_{\mathcal{U}} \otimes \mathrm{Sym}^r(\underline{\omega}^+_{\mathcal{U}} \otimes \underline{\omega}^-_{\mathcal{U}})(C^\infty)$, hence we get a map

$$(20) \qquad \mathbf{d}^r : H^0(\mathrm{M}_{\mathcal{U}}, \underline{\omega}^{(k,l)}_{\mathcal{U}}) \to H^0(\mathrm{M}_{\mathcal{U}}, \underline{\omega}^{(k,l)}_{\mathcal{U}} \otimes \mathrm{Sym}^r(\underline{\omega}^+_{\mathcal{U}} \otimes \underline{\omega}^-_{\mathcal{U}})) \,.$$

From our discussion in Section 5.3, we see that restricting to holomorphic global section on $\mathrm{Sh}_{\mathcal{U}}$ yields a map

$$(21) \qquad \mathbf{d}^r : M_{(k,l)}(\mathcal{U}; R) \to H^0(\mathrm{Sh}_{\mathcal{U}}, s^*_{\mathcal{U}}(\underline{\omega}^{(k,l)}_{\mathcal{U}} \otimes \mathrm{Sym}^r(\underline{\omega}^+_{\mathcal{U}} \otimes \underline{\omega}^-_{\mathcal{U}}))) \,.$$

5.5. Algebraic restrictions.
Let $(V_1, \langle \cdot, \cdot \rangle_1)$ and $(V_2, \langle \cdot, \cdot \rangle_2)$ be two Hermitian K-vector spaces such that $V = V_1 \oplus V_2$, as in Section 2.4. We again write (m_i, n_i) for the signature of V_i (with respect to our fixed choice of embedding $\sigma : K \hookrightarrow \mathbb{C}$), hence $m = m_1 + m_2$ and $n = n_1 + n_2$.

For $i = 1, 2$, let G_i denote the algebraic group $\mathrm{GU}(V_i, \langle \cdot, \cdot \rangle_i)$. We write X_i for the corresponding locally symmetric space, $\mathrm{Sh}_{\mathcal{U}_i}(G_i, X_i)$ for the Shimura variety of level $\mathcal{U}_i \subset G_i(\mathbb{A}_f)$ and so on. Let $s_{\mathcal{U}_i} : \mathrm{Sh}_{\mathcal{U}_i}(G_i, X_i) \hookrightarrow \mathrm{M}_{\mathcal{U}_i}(G_i, X_i)$ denote the canonical

inclusion. Furthermore, the analogue of the algebraic group H for G_i is $H_i = \mathrm{GL}(m_i) \times \mathrm{GL}(n_i)$.

To describe the restrictions in Section 2.4, one needs to consider the intermediate reductive group $G' = \mathrm{G}(\mathrm{U}(V_1) \times \mathrm{U}(V_2))$ whose R-points are given by

$$G'(R) := \{(g_1, g_2) \in G_1(R) \times G_2(R) \mid \nu(g_1) = \nu(g_2)\},$$

for any $\mathbb{Q}$-algebra R.

Proceeding as in Section 5.1, one can again associate a locally symmetric space X' to G' such that (G', X') is a Shimura datum (see [EHLS20, Section 3] for precise details). Thus, we similarly obtain towers of Shimura varieties $\mathrm{Sh}_{\mathcal{U}'}(G', X')$ and moduli spaces $\mathrm{M}_{\mathcal{U}'} = \mathrm{M}_{\mathcal{U}'}(G', X')$ indexed by neat compact open subgroups $\mathcal{U}'$ of $G'(\mathbb{A}_f)$. Let $s_{\mathcal{U}'} : \mathrm{Sh}_{\mathcal{U}'}(G', X') \hookrightarrow \mathrm{M}_{\mathcal{U}'}(G', X')$ denote the canonical inclusion. The analogue of the algebraic group H for G' is $H' = H_1 \times H_2$.

We denote the natural embeddings $G' \hookrightarrow G$ and $G' \hookrightarrow G_1 \times G_2$ by ι and $\iota_{1,2}$ respectively.

Let $\mathcal{U}_i \subset G_i(\mathbb{A}_f)$, $\mathcal{U}' \subset G'(\mathbb{A}_f)$ and $\mathcal{U} \subset G(\mathbb{A}_f)$ be neat open compact subgroups such that $\mathcal{U}' \subset (\mathcal{U}_1 \times \mathcal{U}_2) \cap G'(\mathbb{A}_f)$ and $\mathcal{U}' \subset \mathcal{U} \cap G'(\mathbb{A}_f)$. Then, ι and $\iota_{1,2}$ induce maps

$$\mathrm{M}_{\mathcal{U}'}(G', X') \to \mathrm{M}_{\mathcal{U}}(G, X) \quad \text{and} \quad \mathrm{M}_{\mathcal{U}'}(G', X') \to \mathrm{M}_{\mathcal{U}_1}(G_1, X_1) \times \mathrm{M}_{\mathcal{U}_2}(G_2, X_2),$$

and

$$\mathrm{Sh}_{\mathcal{U}'}(G', X') \to \mathrm{Sh}_{\mathcal{U}}(G, X) \quad \text{and} \quad \mathrm{Sh}_{\mathcal{U}'}(G', X') \to \mathrm{Sh}_{\mathcal{U}_1}(G_1, X_1) \times \mathrm{Sh}_{\mathcal{U}_2}(G_2, X_2),$$

respectively, which we denote ι and $\iota_{1,2}$ again when the level subgroups are clear from context.

From now on, we fix neat open subgroups $\mathcal{U} \subset G(\mathbb{A}_f)$, $\mathcal{U}_1 \subset G_1(\mathbb{A}_f)$ and $\mathcal{U}_2 \subset G_2(\mathbb{A}_f)$ such that $\mathcal{U}' := (\mathcal{U}_1 \times \mathcal{U}_2) \cap G'(\mathbb{A}_f)$ is again a neat open compact subgroup and $\mathcal{U}' \subset \mathcal{U} \cap G'(\mathbb{A}_f)$. To emphasize this choice, we sometimes write $\mathcal{U}_{1,2}$ for $\mathcal{U}'$.

In this situation, the map $\iota_{1,2}$ is in fact an isomorphism at the level of Shimura varieties, i.e. identifies the canonical connected components, hence we obtain

$$\tag{22} \mathrm{Sh}_{\mathcal{U}_{1,2}}(G', X') \xrightarrow{\sim} \mathrm{Sh}_{\mathcal{U}_1}(G_1, X_1) \times \mathrm{Sh}_{\mathcal{U}_2}(G_2, X_2).$$

Write A for the universal abelian variety $A_{\mathcal{U}}^{univ}$ over $\mathrm{M}_{\mathcal{U}}(G, X)$, as in Section 5.2. Similarly, let A_i be the universal abelian variety over $\mathrm{M}_{\mathcal{U}_i}(G_i, X_i)$. Then, by uniqueness, we obtain a canonical identification

$$s_{\mathcal{U}_{1,2}}^* \iota^* A \simeq s_{\mathcal{U}_1}^* A_1 \times s_{\mathcal{U}_2}^* A_2$$

of abelian varieties over $\mathrm{Sh}_{\mathcal{U}_1}(G_1, X_1) \times \mathrm{Sh}_{\mathcal{U}_2}(G_2, X_2)$, using (22).

This identification induces a isomorphisms

$$\tag{23} s_{\mathcal{U}_{1,2}}^* \iota^* \omega_{\mathcal{U}}^{\pm} \xrightarrow{\sim} s_{\mathcal{U}_1}^* \omega_{\mathcal{U}_1}^{\pm} \boxplus s_{\mathcal{U}_2}^* \omega_{\mathcal{U}_2}^{\pm}$$

and

$$\tag{24} s_{\mathcal{U}_{1,2}}^* \iota^* \omega_{\mathcal{U}}^{(k,l)} \xrightarrow{\sim} s_{\mathcal{U}_1}^* \omega_{\mathcal{U}_1}^{(k,l)} \boxtimes s_{\mathcal{U}_2}^* \omega_{\mathcal{U}_2}^{(k,l)},$$

adapting the definitions in Example 5.4 to G, G_1 and G_2 for the latter.

As in Section 5.3, for $i = 1, 2$, let $\rho_{i,\mathrm{std}}^+$ (resp. $\rho_{i,\mathrm{std}}^-$) stand for the m_i-dimensional (resp. n_i-dimensional) standard representation of $\mathrm{GL}(m_i)$ (resp. $\mathrm{GL}(n_i)$).

Again, this provides a basis for $\rho_{i,\mathrm{std}} := \rho_{i,\mathrm{std}}^+ \boxtimes \rho_{i,\mathrm{std}}^-$ and more generally, a basis for $W_{\kappa_i} = \mathbb{S}_{\kappa_i}(\rho_{i,\mathrm{std}})$, for each dominant weight κ_i of H_i. These choices are compatible with the ones made in Section 5.3 via the canonical identity $\rho_{\mathrm{std}} = \rho_{1,\mathrm{std}} \oplus \rho_{2,\mathrm{std}}$ (as representations of $H_1 \times H_2$).

5.6. **Algebraic differential operators.**

5.6.1. *Restriction.* Consider the map

$$\tag{25} \omega_{\mathcal{U}}^{(k,l)} \to \omega_{\mathcal{U}}^{(k,l)} \otimes \mathrm{Sym}^r(\omega_{\mathcal{U}}^+ \otimes \omega_{\mathcal{U}}^-)$$

obtained at the end of Section 5.4. In other words, this is (21) at the level of sheaves on $\mathrm{Sh}_{\mathcal{U}}$.

Remark 5.5. In what follows, we work with (sheaves on) Shimura varieties. To lighten the notation, we omit the pullback via the canonical inclusions $s'_{\mathcal{U}}$, $s_{\mathcal{U}_1}$ and $s_{\mathcal{U}_2}$ from our notation.

Let $f \in M_{(k,l)}(\mathcal{U}, \mathbb{C})$, so $\iota^*(\mathbf{d}^r f)$ is a section of the pullback to $\mathrm{Sh}_{\mathcal{U}_1}(G_1, X_1) \times \mathrm{Sh}_{\mathcal{U}_2}(G_2, X_2)$ of the target of the map (25).

Remark 5.6. In what follows, the level subgroups $\mathcal{U}_1$ and $\mathcal{U}_2$ are fixed. Therefore, to lighten the notation, we replace all subscripts $\mathcal{U}_i$ by i and set $\mathrm{Sh}_i := \mathrm{Sh}_{\mathcal{U}_i}(G_i, X_i)$.

Using (23), (24) and our discussion in Section 5.5, we see that this target is isomorphic to

$$(\underline{\omega}_1^{(k,l)} \boxtimes \underline{\omega}_2^{(k,l)}) \otimes \mathrm{Sym}^r \left((\underline{\omega}_1^+ \boxplus \underline{\omega}_2^+) \otimes (\underline{\omega}_1^- \boxplus \underline{\omega}_2^-) \right)$$

Then, write $(\underline{\omega}_1^+ \boxplus \underline{\omega}_2^+) \otimes (\underline{\omega}_1^- \boxplus \underline{\omega}_2^-)$ as

$$\tag{26} \iota_1^*(\underline{\omega}_1^+ \otimes \underline{\omega}_1^-) \oplus \iota_2^*(\underline{\omega}_2^+ \otimes \underline{\omega}_2^-) \oplus (\underline{\omega}_1^+ \boxtimes \underline{\omega}_2^-) \oplus (\underline{\omega}_1^- \boxtimes \underline{\omega}_2^+)$$

and consider the projection

$$\mathrm{Sym}^r \left((\underline{\omega}_1^+ \boxplus \underline{\omega}_2^+) \otimes (\underline{\omega}_1^- \boxplus \underline{\omega}_2^-) \right) \to \mathrm{Sym}^r \left(\iota_1^*(\underline{\omega}_1^+ \otimes \underline{\omega}_1^-) \oplus \iota_2^*(\underline{\omega}_2^+ \otimes \underline{\omega}_2^-) \oplus (\underline{\omega}_1^+ \boxtimes \underline{\omega}_2^-) \right),$$

where we simply quotient out the last factor of (26).

Therefore, for any integer $r \geq 1$, we have a differential operator

$$\Theta_x^r : \underline{\omega}_1^{(k,l)} \boxtimes \underline{\omega}_2^{(k,l)} \to (\underline{\omega}_1^{(k,l)} \boxtimes \underline{\omega}_2^{(k,l)}) \otimes \mathrm{Sym}^r(\iota_1^*(\underline{\omega}_1^+ \otimes \underline{\omega}_1^-) \oplus \iota_2^*(\underline{\omega}_2^+ \otimes \underline{\omega}_2^-) \oplus (\underline{\omega}_1^+ \boxtimes \underline{\omega}_2^-)),$$

on $\mathrm{Sh}_1 \times \mathrm{Sh}_2$.

Similarly, the projection corresponding to omitting $\underline{\omega}_{\mathcal{U}_1}^+ \boxtimes \underline{\omega}_{\mathcal{U}_2}^+$ yields an operator

$$\Theta_y^r : \underline{\omega}_1^{(k,l)} \boxtimes \underline{\omega}_2^{(k,l)} \to (\underline{\omega}_1^{(k,l)} \boxtimes \underline{\omega}_2^{(k,l)}) \otimes \mathrm{Sym}^r(\iota_1^*(\underline{\omega}_1^+ \otimes \underline{\omega}_1^-) \oplus \iota_2^*(\underline{\omega}_2^+ \otimes \underline{\omega}_2^-) \oplus (\underline{\omega}_1^- \otimes \underline{\omega}_2^+)),$$

on $\mathrm{Sh}_1 \times \mathrm{Sh}_2$.

Remark 5.7. The operators Θ_x^r and Θ_y^r are not exactly the algebraic analogues of d_x^r and d_y^r from (5). In terms of coordinates, Θ_r^x corresponds to taking $y = 0$ and collecting all derivatives with respect to (τ_1, τ_2, x) of order r, using the notation of Section 2.5.

5.6.2. *Algebraic vanishing conditions.* We focus on Θ_x^r first. By writing

$$\mathrm{Sym}^r\left(\iota_1^*(\underline{\omega}_1^+ \otimes \underline{\omega}_1^-) \oplus \iota_2^*(\underline{\omega}_2^+ \otimes \underline{\omega}_2^-) \oplus (\underline{\omega}_1^+ \otimes \underline{\omega}_2^-)\right)$$

$$=\bigoplus_{s=0}^{r}\mathrm{Sym}^{r-s}\left(\iota_1^*(\underline{\omega}_1^+ \otimes \underline{\omega}_1^-) \oplus \iota_2^*(\underline{\omega}_2^+ \otimes \underline{\omega}_2^-)\right) \otimes \mathrm{Sym}^s\left(\underline{\omega}_1^+ \otimes \underline{\omega}_2^-\right),$$

one readily sees that f satisfies the conditions of Proposition 4.3 if and only if it satisfies

Condition 5.8. The section $\Theta_x^r f$ is actually a section of the subbundle

$$(\underline{\omega}_1^{(k,l)} \boxtimes \underline{\omega}_2^{(k,l)}) \otimes \mathrm{Sym}^r\left(\underline{\omega}_1^+ \boxtimes \underline{\omega}_2^-\right)$$

of $\mathrm{Sym}^r\left(\iota_1^*(\underline{\omega}_1^+ \otimes \underline{\omega}_1^-) \oplus \iota_2^*(\underline{\omega}_2^+ \otimes \underline{\omega}_2^-) \oplus (\underline{\omega}_1^+ \otimes \underline{\omega}_2^-)\right)$.

Let $\mathrm{M}_{(k,l)}^{x,r}(\mathcal{U};\mathbb{C})$ be the subspace of forms $f \in \mathrm{M}_{(k,l)}^{x,r}(\mathcal{U};\mathbb{C})$ satisfying Condition 5.8. Similarly, let $\mathrm{M}_{(k,l)}^{y,r}(\mathcal{U};\mathbb{C})$ be the subspace of forms f satisfying

Condition 5.9. The section $\Theta_y^r f$ is actually a section of the subbundle

$$(\underline{\omega}_1^{(k,l)} \boxtimes \underline{\omega}_2^{(k,l)}) \otimes \mathrm{Sym}^r\left(\underline{\omega}_1^- \boxtimes \underline{\omega}_2^+\right)$$

of $\mathrm{Sym}^r\left(\iota_1^*(\underline{\omega}_1^+ \otimes \underline{\omega}_1^-) \oplus \iota_2^*(\underline{\omega}_2^+ \otimes \underline{\omega}_2^-) \oplus (\underline{\omega}_1^- \otimes \underline{\omega}_2^+)\right)$.

5.6.3. *Algebraic differential operators associated to partitions.* Note that the algebraic analogue of Equation (7) states

$$(27) \qquad \mathrm{Sym}^r\left(\underline{\omega}_1^{\pm} \boxtimes \underline{\omega}_2^{\mp}\right) = \bigoplus_\lambda \mathbb{S}^\lambda\left(\underline{\omega}_1^{\pm}\right) \boxtimes \mathbb{S}^\lambda\left(\underline{\omega}_2^{\mp}\right),$$

where the sum runs over all partitions λ of r.

For any such partition λ of r, let $\rho_{i,(k,l)}^{+,\lambda}$ and $\rho_{i,(k,l)}^{-,\lambda}$ denote the representations

$$(\mathrm{det}_{\mathrm{GL}(m_i)}^k \otimes \mathbb{S}^\lambda(\rho_{\mathrm{std}})) \boxtimes \mathrm{det}_{\mathrm{GL}(n_i)}^l,$$

and

$$\mathrm{det}_{\mathrm{GL}(m_i)}^k \boxtimes (\mathrm{det}_{\mathrm{GL}(n_i)}^l \otimes \mathbb{S}^\lambda(\rho_{\mathrm{std}})).$$

of H_i, for $i = 1, 2$. In other words, $\rho_{i,(k,l)}^{\pm,\lambda}$ is the representation associated to the automorphic vector bundle

$$\omega_{\mathcal{U}_i}^{(k,l)} \otimes \mathbb{S}^\lambda(\underline{\omega}_{\mathcal{U}_i}^{\pm}).$$

Let $\kappa_{i,(k,l)}^{\pm,\lambda}$ denote the highest weight of $\rho_{i,(k,l)}^{\pm,\lambda}$. Then, using the discussion of Section 5.3, one has a natural identification

$$(28)$$
$$H^0(\mathrm{Sh}_1 \times \mathrm{Sh}_2, (\underline{\omega}_1^{(k,l)} \otimes \mathbb{S}^\lambda(\underline{\omega}_1^{\pm})) \boxtimes (\underline{\omega}_2^{(k,l)} \otimes \mathbb{S}^\lambda(\underline{\omega}_2^{\mp}))) \xrightarrow{\sim} \mathrm{M}_{\kappa_{1,(k,l)}^{\pm,\lambda}}(\mathcal{U}_1;\mathbb{C}) \otimes \mathrm{M}_{\kappa_{2,(k,l)}^{\mp,\lambda}}(\mathcal{U}_2;\mathbb{C})$$

Then, the following is a reformulation of Theorem 2.14 and Section 4 in the algebraic language of the current section.

Theorem 5.10. *Fix an $r \geq 1$ and a partition λ of r. The composition of Θ_x^r and Θ_y^r with the projection to the λ-component in Equation* (27), *respectively, yields differential operators*

$$\Theta_{x,\lambda}^r : \mathrm{M}_{(k,l)}^{x,r}(\mathcal{U}; \mathbb{C}) \to \mathrm{M}_{\kappa_{1,(k,l)}^{+,\lambda}}(\mathcal{U}_1; \mathbb{C}) \otimes \mathrm{M}_{\kappa_{2,(k,l)}^{-,\lambda}}(\mathcal{U}_2; \mathbb{C}),$$

and

$$\Theta_{y,\lambda}^r : \mathrm{M}_{(k,l)}^{y,r}(\mathcal{U}; \mathbb{C}) \to \mathrm{M}_{\kappa_{1,(k,l)}^{-,\lambda}}(\mathcal{U}_1; \mathbb{C}) \otimes \mathrm{M}_{\kappa_{2,(k,l)}^{+,\lambda}}(\mathcal{U}_2; \mathbb{C}),$$

via the identification (28)*. These differential operators respect subspaces of cusp forms.*

Remark 5.11. Naturally, these differential operators are defined over any $\mathbb{C}$-algebra R, by simply replacing $\mathbb{C}$ with such a ring R in all of the above. In fact, it is not to hard to see that they are well defined over K (and any K-algebra) since the Gauss–Manin connection and the Kodaira–Spencer isomorphism in Section 5.4 both make sense over K. We do not emphasize this point of view in this paper to focus on applications to $\mathbb{C}$-valued modular forms as in Section 3.

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THOMAS L. BROWNING, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, BERKELEY, EVANS HALL, BERKELEY, CA 94720-3840, USA
   Email address: tb1004913@berkeley.edu

PAVEL ČOUPEK, DEPARTMENT OF MATHEMATICS, MICHIGAN STATE UNIVERSITY, WELLS HALL, 619 RED CEDAR ROAD, EAST LANSING, MI 48824, USA
   Email address: coupekpa@msu.edu

ELLEN EISCHEN, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF OREGON, FENTON HALL, EUGENE, OR 97403-1222, USA
   Email address: eeischen@uoregon.edu

CLAIRE FRECHETTE, DEPARTMENT OF MATHEMATICS, BOSTON COLLEGE, MALONEY HALL, CHESTNUT HILL, MA 02467-3806, USA
   Email address: frechecl@bc.edu

SERIN HONG, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ARIZONA, 617 N SANTA RITA AVE, TUCSON, AZ 85721, USA
   Email address: serinh@arizona.edu

SI YING LEE, DEPARTMENT OF MATHEMATICS, STANFORD UNIVERSITY, BUILDING 380, STANFORD, CA 94305, USA
   Email address: leesy1@stanford.edu

DAVID MARCIL, DEPARTMENT OF MATHEMATICS, COLUMBIA UNIVERSITY, 2990 BROADWAY, NEW YORK, NY 10027, USA
   Email address: d.marcil@columbia.edu