Modular forms on Hilbert modular varieties

STAGE

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Review

Kummer Congruences

Let B_n be the n-th Bernoulli number, and m, n be positive even integers with $m \equiv n \pmod{p^{a-1}(p-1)}$ and $n \not\equiv 0 \pmod{p-1}$. Then $\frac{B_m}{m}$ and $\frac{B_n}{n}$ are in $\mathbb{Z}_p$ and

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- This allows us to define a p-adic L-function interpolating values of the zeta function - namely the Kubota-Leopoldt p-adic L function
- We can construct the p-adic L-function from p-adic measures i.e. for all $a \in \mathbb{Z}_p^\times$ there exists some measure $\mu^{(a)}$ on $\mathbb{Z}_p^\times$ such that

$$\int_{\mathbb{Z}_p^\times} x^k d\mu^{(a)} = (1-a^{k+1})(1-p^k)\zeta(-k)$$

p-adic L-functions

- We can also construct p-adic L functions associated to Dirichlet L-functions. Let χ be an even Dirichlet character of conductor p^n for some $n \geq 0$, we have

$$\int_{\mathbb{Z}_p^\times} \chi(x)x^k d\mu^{(a)} = (1-\chi(a)a^{k+1})(1-\chi(p)p^k)L(-k,\chi)$$

- If we fix k, we can view the above measure $\mu^{(a)}$ as interpolating $L(-k,\chi)$ for all Dirichlet characters χ as above

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- If we fix k, we can view the above measure $\mu^{(a)}$ as interpolating $L(-k,\chi)$ for all Dirichlet characters χ as above
- We have $\mathbb{Z}_p^\times = Gal(\mathbb{Q}(p^\infty)/\mathbb{Q})$, where $\mathbb{Q}(p^\infty)$ is the maximal abelian extension of $\mathbb{Q}$ unramified outside of p
- For any number field $F/\mathbb{Q}$, the p-adic L-function should be a measure on $Gal(F(p^\infty)/F)$, where $F(p^\infty)$ is the maximal abelian extension of F unramified outside of p

Overview of Katz

- Fix a CM field L with ordinary CM type Σ with respect to some embedding $\bar{L} \hookrightarrow C_p$: for any $\sigma \in \Sigma, \tau \in \bar{\Sigma}$, the p-adic valuation on L induced by the embeddings

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  give rise to distinct p-adic valuations
- The main goal of Katz's paper is to construct a p-adic L-function which interpolates the values $L(0, \chi)$ for some Hecke grossencharacters χ, i.e. we want some measure μ such that

$$\int_{Gal(L(p^\infty)/L)} \chi d\mu \text{``}=\text{''} L(0, \chi)$$

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- Katz also shows this p-adic L-function satisfies a functional equation
- Key idea: interpolating family of p-adic Hilbert modular forms obtained via differential operators from Eisenstein series

Hilbert-Blumenthal Abelian Varieties

- Fix a totally real field K of degree g over $\mathbb{Q}$
- Let $\mathcal{D}^{-1}$ be the inverse different of K, i.e.

$$\mathcal{D}^{-1} := \{x \in F : tr(xy) \in \mathbb{Z} \text{ for all } y \in O_K\}$$

- Let $O := O_K$ be the ring of integers of K
- Let $\mathfrak{c}$ be any fractional ideal of K

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Definition

A Hilbert-Blumenthal abelian variety (HBAV) over a scheme S is a g-dimensional scheme X/S with a map $\iota : O \hookrightarrow End_{O_S}(X)$ such that locally on S, the $O \otimes O_S$-module $Lie(X)$ is free of rank 1

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- Let $X \otimes_O \mathfrak{c}$ be the abelian scheme such that for all schemes S'/S, we have
$$(X \otimes \mathfrak{c})(S') = X(S') \otimes_O \mathfrak{c}$$
- This gives a natural O-linear map
$$\mathfrak{c} \hookrightarrow Hom_O(X, X \otimes \mathfrak{c})$$

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Definition

A $\mathfrak{c}$-polarization is an isomorphism $\lambda : X^\vee \to X \otimes_O \mathfrak{c}$ under which the symmetric elements of $Hom_O(X, X^\vee)$ corresponds to (image of) $\mathfrak{c}$, and the polarizations in $Hom_O(X, X^\vee)$ correspond to $\mathfrak{c}^+$, the cone in $\mathfrak{c}$ of totally positive elements

- The symmetric elements of $Hom_O(X, X^\vee)$ are the maps

$$\{f : X \to X^\vee : f \circ \iota(r) = \iota^\vee(r) \circ f \text{ for all } r \in O\}$$

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- Let $\underline{\omega}_{X/S} := H^0(X, \Omega^1_{X/S})$, this is dual to $Lie(X/S)$, and both are $O \otimes O_S$-modules
- We can upgrade the isomorphism $Lie(X) \otimes_{O_S} \underline{\omega} \xrightarrow{\sim} O_S$ to an $O \otimes O_S$-isomorphism

$$Lie(X) \otimes_{O \otimes O_S} \underline{\omega} \xrightarrow{\sim} \mathcal{D}^{-1} \otimes O_S$$

such that the composition

$$Lie(X) \otimes_{O_S} \underline{\omega} \twoheadrightarrow Lie(X) \otimes_{O \otimes O_S} \underline{\omega} \xrightarrow{\sim} \mathcal{D}^{-1} \otimes O_S \xrightarrow{tr \otimes 1} O_S$$

gives the isomorphism above
- Every $O \otimes O_S$-basis element ω gives an isomorphism

$$Lie(X) \xrightarrow{\sim} \mathcal{D}^{-1} \otimes O_S$$

Moduli spaces of HBAV

Definition

A $\Gamma_{0,0}(N)$-structure on a HBAV X/S is an O-linear homomorphism

$$i : \mathcal{D}^{-1} \otimes_{\mathbb{Z}} \mu_N \hookrightarrow X$$

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Theorem

Let $\mathcal{M}(\mathfrak{c}, N)$ be the moduli space of $\mathfrak{c}$-polarized HBAVs with $\Gamma_{0,0}(N)$ level structure. Then $\mathcal{M}(\mathfrak{c}, N)$ is an algebraic stack over $\mathbb{Z}$ which is smooth of relative dimension g. Moreover, for $N \geq 4$, the moduli problem is rigid, and hence $\mathcal{M}(\mathfrak{c}, N)$ is represented by a scheme.

Hence for $N \geq 4$ we have a universal object

$$(X_{univ}, \lambda_{univ}, i_{univ}) \xrightarrow{\pi} \mathcal{M}(\mathfrak{c}, N)$$

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- Since $Lie(X)$ gives a universal covering of X, we have the short exact sequence

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- A $\mathfrak{c}$-polarization λ on X corresponds exactly to an alternating O-bilinear form

$$\mathcal{L} \times \mathcal{L} \to \mathcal{D}^{-1}\mathfrak{c}^{-1}$$

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  given as $\langle u, v \rangle = \frac{Im(\bar{u}v)}{A}$, for some A
- $\Gamma_{0,0}(N)$-structure corresponds to an injective O-linear map

$$i : \mathcal{D}^{-1} \otimes \mathbb{Z}/n\mathbb{Z} \hookrightarrow \mathcal{L} \otimes \mathbb{Z}/n\mathbb{Z}$$

HBAVs over $\mathbb{C}$

Let $\mathfrak{a}, \mathfrak{b}$ be fractional ideals of F such that $\mathfrak{c} = \mathfrak{a}\mathfrak{b}^{-1}$. We define the lattice

$$\mathcal{L}_{\mathfrak{a},\mathfrak{b}}(\tau) := 2\pi i(\mathcal{D}^{-1}\mathfrak{a}^{-1} \cdot 1 \oplus \mathfrak{b} \cdot \tau)$$

A polarization on $\mathcal{L}_{\mathfrak{a},\mathfrak{b}}(\tau)$ can be described by an alternating pairing:

$$\langle 2\pi i(a + b\tau), 2\pi i(c + d\tau)\rangle = ad - bc$$

(In this case $A = 4\pi Im(\tau)$)

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- Every $\mathfrak{c}$-polarized HBAVs is isomorphic to $\mathcal{L}_{\mathfrak{a},\mathfrak{b}}(\tau)$ for some τ
- $\Gamma_{0,0}(N)$ level structure is determined by an isomorphism ϵ

$$O \otimes \mathbb{Z}/n\mathbb{Z} \xrightarrow{\epsilon} \mathfrak{a}^{-1} \otimes \mathbb{Z}/n\mathbb{Z}$$

  such that

$$\mathcal{D}^{-1} \otimes \mathbb{Z}/n\mathbb{Z} \xrightarrow{\sim} \mathcal{D}^{-1}\mathfrak{a}^{-1} \otimes \mathbb{Z}/n\mathbb{Z} \xrightarrow{2\pi i \cdot} \mathcal{L}_{\mathfrak{a},\mathfrak{b}}(\tau) \otimes \mathbb{Z}/n\mathbb{Z}$$

$\mathcal{M}(\mathfrak{c}, N)_{\mathbb{C}}$

- For any 2 fractional ideals $\mathfrak{m}, \mathfrak{n}$, let

$$SL(\mathfrak{m} \oplus \mathfrak{n}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, d \in O, b \in \mathfrak{m}^{-1}\mathfrak{n}, c \in \mathfrak{m}\mathfrak{n}^{-1}, det = 1 \right\}$$

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$$\left. c \in N\mathcal{D}\mathfrak{a}^{-1}\mathfrak{b}^{-1} \right\}$$

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- $\mathcal{M}(\mathfrak{c}, N)_{\mathbb{C}}$ is isomorphic to $H^g/\Gamma_{0,0}(N)$

Structure of $\mathcal{M}(\mathfrak{c}, N)$

Theorem (Ribet)

The geometric fibres of $\mathcal{M}(\mathfrak{c}, N)$ over $Spec(\mathbb{Z})$ are all geometrically irreducible.

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Sketch of proof:

- The generic fiber is irreducible (smooth and connected)
- Let $(N, p) = 1$. Every HBAV over a field of characteristic p with $\Gamma_{0,0}(p^n)$-structure for some $n \geq 1$ is an ordinary abelian variety. We thus have map

$$f : \mathcal{M}(\mathfrak{c}, Np^n)_{\mathbb{F}_p} \to \mathcal{M}(\mathfrak{c}, N)^{ord}_{\mathbb{F}_p}$$

- If $N \geq 4$, the fibers are principal homogenous spaces under $(O/p^n O)^\times$
- $\mathcal{M}(\mathfrak{c}, N)_{\mathbb{F}_p}$ is irreducible, and $Y := \mathcal{M}(\mathfrak{c}, N)^{ord}_{\mathbb{F}_p}$ is an open dense subset

Theorem (Ribet)

The geometric fibres of $\mathcal{M}(\mathfrak{c}, N)$ over $Spec(\mathbb{Z})$ are all geometrically irreducible.

- To show that $f^{-1}(Y)$ is geometrically irreducible, we have to show the induced monodromy representation

$$\chi : \pi_1(Y \otimes \bar{\mathbb{F}}_p) \to (O/p^n O)^\times$$

is surjective.

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is surjective.

- It then suffices to construct an ordinary abelian variety $\mathcal{A} \in Y$ over $\mathbb{F}_{p^k}$ for sufficiently large k such that

$$\pi_1(Spec(\mathbb{F}_{p^k})) \to \pi_1(Y \otimes \bar{\mathbb{F}}_p) \to (O/p^n O)^\times$$

is surjective, and we observe that $Frob_{p^k}$ is a topological generator of $\pi_1(Spec(\mathbb{F}_{p^k}))$, so it suffices to check the action of Frobenius on $\mathcal{D}^{-1} \otimes \mu_{p^n} \hookrightarrow A[p^n]$.

$\mathfrak{c}$-Hilbert Modular Forms

- Fix a ring R_0
- Let χ be a character of $Res_{O/\mathbb{Z}}\mathbb{G}_m$. Concretely, this as a map $(O \otimes R)^\times$ to $R^\times$ for all rings R/R_0.

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- If R_0 contains $O_{\tilde{K}}$, where $\tilde{K}$ is a normal closure of K, $Res_{O/\mathbb{Z}}\mathbb{G}_m$ splits, then χ is given by a tuple $\mathbf{k} = (k_1, \ldots, k_g) \in \mathbb{Z}^g$.

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Definition

A $\mathfrak{c}$-HMF of weight χ defined over R_0 on $\Gamma_{0,0}(N)$ is a rule f which assigns to every $\mathfrak{c}$-polarized HBAV over R with a nowhere vanishing differential ω and $\Gamma_{0,0}(N)$ structure i, an element $f(X, \lambda, \omega, i) \in R$ such that

1. $f(X, \lambda, \omega, i)$ depends only on the R-isomorphism class of (X, λ, ω, i)
2. f commutes with base change
3. For all $a \in (O \otimes R)^\times$,

$$f(X, \lambda, a^{-1}\omega, i) = \chi(a) f(X, \lambda, \omega, i)$$

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- When $N \geq 4$, we let $\underline{\omega} = \pi_*(\Omega^1_{\mathcal{M}/\mathbb{Z}})$, and $\underline{\omega}(\chi)$ be the extension of the structure group of $\underline{\omega}$ by χ
- If R_0 contains $O_{\tilde{K}}$, then the O-action on $\underline{\omega}$ induces a decomposition $\underline{\omega} = \bigoplus \underline{\omega}_i$, and $\underline{\omega}(\chi) = \bigotimes \underline{\omega_i}^{k_i}$

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- $\mathfrak{c}$-Hilbert Modular Forms over R_0 are thus elements of $H^0(\mathcal{M}(\mathfrak{c}, N)_{R_0}, \underline{\omega}(\chi))$

Analogue of Tate curve

- Let S be the set of g linearly independent $\mathbb{Q}$-linear forms l_i which map the totally positive elements of K to positive rational numbers
- Let $\mathbb{Z}[[\mathfrak{ab}, S]]$ be the ring of all formal series

$$\sum_{\substack{\alpha \in \mathfrak{ab} \\ l_i(\alpha) > 0 \text{ for all } i}} a_\alpha q^\alpha \qquad a_\alpha \in \mathbb{Z}$$

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  and $\mathbb{Z}((\mathfrak{ab}, S))$ the ring obtained from $\mathbb{Z}[[\mathfrak{ab}, S]]$ by inverting all q^α, for α the totally positive elements
- Consider the g-dimensional torus $\mathbb{G}_m \otimes \mathcal{D}^{-1}\mathfrak{a}^{-1}$. We want to construct a subgroup given by $\mathfrak{b}$, i.e. an O_K-linear map $q : \mathfrak{b} \to \mathbb{G}_m \otimes \mathcal{D}^{-1}\mathfrak{a}^{-1}$
- It suffices to construct a $\mathbb{Z}$-linear map $\mathfrak{ab} \to \mathbb{G}_m$, so let $\alpha \mapsto q^\alpha$

Analogue of Tate curve

We hence obtain a rigid analytic HBAV given by

$$\mathbb{G}_m \otimes \mathcal{D}^{-1}\mathfrak{a}^{-1}/q(\mathfrak{b})$$

which algebrizes to a HBAV over $\mathbb{Z}((\mathfrak{a}\mathfrak{b}, S))$ which we denote by $Tate_{\mathfrak{a},\mathfrak{b}}(q)$

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We have a canonical polarization λ_{can} given by the isomorphism induced by

$$Tate_{\mathfrak{a},\mathfrak{b}}(q)^\vee \xrightarrow{\sim} Tate_{\mathfrak{b},\mathfrak{a}}(q) \xrightarrow{\sim} Tate_{\mathfrak{a},\mathfrak{b}}(q) \otimes \mathfrak{c}$$

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Moreover, we observe that we have an injection

$$\mathcal{D}^{-1}\mathfrak{a}^{-1} \otimes \mu_N \hookrightarrow Tate_{\mathfrak{a},\mathfrak{b}}[N]$$

so fixing an isomorphism $\varepsilon : O/NO \to \mathfrak{a}^{-1}/N\mathfrak{a}^{-1}$ gives us the desired embedding

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If $\mathfrak{a}$ is prime to N, then both $N, \mathfrak{a}$ have the same N-adic completions, so we have a canonical isomorphism

q-expansions

The Lie algebra of $Tate_{\mathfrak{a},\mathfrak{b}}$ is canonically given by

$$Lie(\mathbb{G}_m \otimes \mathcal{D}^{-1}\mathfrak{a}^{-1}) = \mathcal{D}^{-1}\mathfrak{a}^{-1} \otimes \mathbb{Z}((\mathfrak{a}\mathfrak{b}, s)).$$

If we have an isomorphism $j : \mathfrak{a}^{-1} \otimes R_0 \to O_K \otimes R_0$, then we have an isomorphism $Lie(Tate_{\mathfrak{a},\mathfrak{b}}(q)) \simeq \mathcal{D}^{-1} \otimes R_0((\mathfrak{a}\mathfrak{b}, s))$, which gives us an element $\omega_{\mathfrak{a}}(j) \in \underline{\omega}$

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The Lie algebra of $Tate_{\mathfrak{a},\mathfrak{b}}$ is canonically given by

$$Lie(\mathbb{G}_m \otimes \mathcal{D}^{-1}\mathfrak{a}^{-1}) = \mathcal{D}^{-1}\mathfrak{a}^{-1} \otimes \mathbb{Z}((\mathfrak{a}\mathfrak{b}, s)).$$

If we have an isomorphism $j : \mathfrak{a}^{-1} \otimes R_0 \to O_K \otimes R_0$, then we have an isomorphism $Lie(Tate_{\mathfrak{a},\mathfrak{b}}(q)) \simeq \mathcal{D}^{-1} \otimes R_0((\mathfrak{a}\mathfrak{b}, s))$, which gives us an element $\omega_{\mathfrak{a}}(j) \in \underline{\omega}$

- Given $\mathfrak{c}$-HMF f, choose isomorphisms $\varepsilon : O/NO \to \mathfrak{a}^{-1}/N\mathfrak{a}^{-1}$ and $j : \mathfrak{a}^{-1} \otimes R_0 \to O_K \otimes R_0$, the q-expansion of f at the cusp $(\mathfrak{a}, \mathfrak{b}, j, i(\varepsilon))$ is the value

$$f(Tate_{\mathfrak{a},\mathfrak{b}}(q), \lambda_{can}, \omega_{\mathfrak{a}}(j), i(\varepsilon)) \in R_0((\mathfrak{a}\mathfrak{b}, S))$$

i.e.

$$f(Tate_{\mathfrak{a},\mathfrak{b}}(q), \lambda_{can}, \omega_{\mathfrak{a}}(j), i(\varepsilon)) = \sum_{\alpha \in \mathfrak{a}\mathfrak{b}} a(f, \alpha) q^{\alpha},$$

for some $a(f, \alpha) \in R_0$

q-expansion principle

Proposition

If the q-expansion of any HMF f is zero at any cusp, then $f = 0$

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Corollary

If f is a HMF defined over R such that the q-expansion coefficients of f at one cusp all lie in R_0, then there is some HMF defined over R_0 which gives rise to f via base-change.

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Sketch of proof: Argument is similar to that for modular curves: we know the form on an open neighborhood of the cusp, and since we have Ribet's irreducibility result, the form is identically 0

q-expansion for HMFs over $\mathbb{C}$

- By GAGA and the q-expansion principle, we see that giving any $\mathfrak{c} - HMF$ over $\mathbb{C}$ is equivalent to giving a holomorphic function f on $\mathcal{M}(\mathfrak{c}, N)$ transforming by χ under the action of $a \in (K \otimes \mathbb{C})^\times$, which is meromorphic at the cusps

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- Since f is invariant under translation by an element of $\mathcal{D}^{-1}\mathfrak{a}^{-1}\mathfrak{b}^{-1}$, we can also write the q-expansion for HMFs defined over $\mathbb{C}$ as

$$f = \sum_{\alpha \in \mathfrak{ab}} a_\alpha \exp(2\pi i Tr(\alpha\tau))$$

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- If $g > 1$, such f is holomorphic at the cusps (Koecher's principle)

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$a_\alpha = 0$ unless $\alpha = 0$ or α is totally real

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- Assume there exists some $\alpha_0 \in \mathfrak{ab}$ not totally positive with $a_{\alpha_0} \neq 0$
- Choose an embedding $\tau_0 : K \hookrightarrow \mathbb{R}$ such that $\tau_0(\alpha_0) < 0$
- By Dirichlet's Unit Theorem, there exists some $\epsilon \in O^{\times,+}$ such that $\tau(\epsilon) < 1$ for all $\tau \neq \tau_0$ $\tau_0(\epsilon) > 1$
- Consider the subseries

$$\sum_{n\in\mathbb{N}} a_{\alpha\epsilon^{2n}} e^{2\pi i \operatorname{Tr}(\alpha_0\epsilon^{2n} z)}$$

- Since $\begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon^{-1} \end{pmatrix} \in \Gamma_{0,0}(N)$, we have

$$a_{\alpha\epsilon^2} = a_\alpha \prod_i \tau_i(\epsilon)^{k_i}$$

- Take $z = (i, \ldots, i)$, and observe that $\sum_{n\in\mathbb{N}} e^{-2\pi \operatorname{Tr}(\alpha_0\epsilon^{2n})}$ diverges

Kodaira-Spencer Isomorphism

Similar to the case of modular curves, we have the following Kodaira-Spencer isomorphism

$$\Omega^1_{\mathcal{M}/\mathbb{Z}} \xrightarrow{\sim} \underline{\omega}^{\otimes 2} \otimes_{O_K} \mathfrak{c}^{-1}$$

given as follows. We have the SES

$$0 \to \underline{\omega} \to H^1_{dR} \to Lie(X^{univ\vee}) \to 0.$$

Given any derivation D, we can define the map

$$KS(D) : \underline{\omega} \to H^1_{dR} \xrightarrow{\nabla(D)} H^1_{dR} \to Lie(X^{univ\vee}) \simeq Lie(X^{univ}) \otimes \mathfrak{c}$$

This induces an isomorphism between the tangent space and $Hom_{O \otimes O_{\mathcal{M}}}(\underline{\omega}, Lie \otimes_O \mathfrak{c}) \simeq Lie^{\otimes 2} \otimes_O \mathcal{D}^{-1}\mathfrak{c}$, the dual of which is the map above

Kodaira Spencer Map for $Tate_{\mathfrak{a},\mathfrak{b}}$

- For every $\gamma \in \mathcal{D}^{-1}\mathfrak{a}^{-1}\mathfrak{b}^{-1}$, we have a derivation

$$D(\gamma)(\sum a_\alpha q^\alpha) = \sum tr(\alpha\gamma) a_\alpha q^\alpha$$

- The Kodaira-Spencer map in this case is a map

$$KS : Der(\mathbb{Z}((\mathfrak{ab}, s))) \to Lie^{\otimes 2} \otimes_O \mathcal{D}^{-1}\mathfrak{c} = \mathcal{D}^{-1}\mathfrak{a}^{-1}\mathfrak{b}^{-1} \otimes \mathbb{Z}((\mathfrak{ab}, s))$$

and in fact $KS(D(\gamma))$ maps to $\gamma \otimes 1$