UNRAMIFIEDNESS OF WEIGHT 1 GALOIS REPRESENTATIONS FOR HILBERT MODULAR VARIETIES

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Abstract. We show local-global compatibility at p for Hecke eigenclasses in higher coherent cohomology in weight one for Hilbert modular varieties at an unramified prime, assuming that the residual degree is odd.

1. Introduction

Let p be prime number. The Serre weight conjecture for modular forms, specialized to the case of weight one, says the following:

Conjecture 1.0.1. *Let*

$$\rho : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$$

be a continuous, irreducible, and odd Galois representation, which is moreover unramified at p. Then, there exists a mod p modular eigenform f of weight one such that ρ_f, the Galois representation attached to f by Deligne-Serre [DS74] *is $\rho_f \simeq \rho$. Moreover, this induces an equivalence between weight one mod p modular eigenforms and odd irreducible unramified mod p Galois representations.*

This modularity result (at least when $p > 2$) was proved in the works of Gross [Gro90], Edixhoven [Edi92], Coleman-Voloch [CV92] and Wiese [Wie14].

The Serre weight conjectures have a natural generalization to the case of Hilbert modular varieties associated to a totally real field F. This was stated in the case of regular weight by Buzzard-Diamond-Jarvis [BDJ10], and proved (assuming modularity of the residual representation) by Gee-Liu-Savitt [GLS15]. In the case of partial weight one, the same weight receipe in [BDJ10] also gives us a conjectural relation between mod p Hilbert modular forms which are of partial weight 1 and mod p representations of $\mathrm{Gal}(\bar{F}/F)$ which are unramified at some prime $\mathfrak{p}$ of F above p.

More precisely, let F be a totally real field of degree d, and let $p > 2$ be a prime unramified in F. Let Sh be the Hilbert modular variety over $\mathbb{Z}_p$ defined using the group $\mathrm{Res}_{F/\mathbb{Q}}\mathrm{GL}_2$, with level hyperspecial level at p, and we let $\mathrm{Sh}^{\mathrm{tor}}$ be an appropriately chosen smooth toroidal compactification. Let Σ be the set of embeddings $\tau : F \hookrightarrow \mathbb{R}$. We let $\mathfrak{p}$ denote a prime of F above p. Let $\bar{\rho}$ be an irreducible mod p Galois representation associated to a Hecke eigensystem appearing in $H^\bullet(\mathrm{Sh}^{\mathrm{tor}}_{\mathbb{F}_p}, \omega^{\underline{\kappa}})$ where $\underline{\kappa} = ((\kappa_\tau)_\tau, w)$ is a paritious weight. This Galois representation was constructed by Emerton-Reduzzi-Xiao [ERX17b].

Now, we fix an isomorphism $\iota : \bar{\mathbb{Q}}_p \simeq \mathbb{C}$, and we let $\Sigma_\mathfrak{p}$ denote the set of embeddings τ which, under ι, induce the prime $\mathfrak{p}$. Let $k_\mathfrak{p}$ denote the residue field of O_F corresponding to $\mathfrak{p}$. We assume that we are of partial weight 1 for $\mathfrak{p}$, ie. $\underline{\kappa}$ satisfies $\kappa_\tau = 1$ for all $\tau \in \Sigma_\mathfrak{p}$, and $w = -1$. The main theorem of this paper is the following:

Theorem 1.0.2. *If the residual degree* $[k_{\mathfrak{p}} : \mathbb{F}_p]$ *is odd, then any absolutely irreducible representation* $\bar{\rho}$ *appearing in* $H^i(\mathrm{Sh}^{\mathrm{tor}}_{\mathbb{F}_p}, \omega^{\underline{\kappa}})$ *is unramified at* $\mathfrak{p}$.

In fact, we show a stronger result that regardless of the residual degree, as long as the cohomology class is not of a 'middle degree' situation, the desired statement about the Galois representation holds. This confirms (part of) a conjecture of Emerton-Reduzzi-Xiao [ERX17a, Conj 5.15], under the same assumptions.

Other than the work of Emerton, Reduzzi and Xiao, there has been previous work on this conjecture by Dimitrov and Wiese [DW18], Deo, Dimitrov and Wiese [DDW24], and de Maria [de 20]. All these three papers have focused on degree zero coherent cohomology, and the only previously known result of unramifiedness for higher coherent cohomology was in [ERX17a], which dealt with the case of Hilbert modular surfaces, though we note that their methods for getting the result in cohomological degree one are completely different from those in this paper. However, coherent cohomology in partial weight one is expected to have torsion in many consecutive degrees, so studying degree zero coherent cohomology is insufficient, necessitating a different approach.

In the following, for notational simplicity, we will assume that p is inert. To motivate our approach, we now recall the main arguments used in the degree zero case to prove that the Galois representation is unramified. One can divide the argument into two steps: the first is to show that the cohomological class is ordinary in the sense of Hida, namely it lies in some space of automorphic forms on which the U_p-operator acts idempotently. More precisely, in degree zero we see that multiplication by the Hasse invariant gives us an injective map

$$h : H^0(\mathrm{Sh}, \omega^{\mathbf{1}}) \hookrightarrow H^0(\mathrm{Sh}, \omega^{\mathbf{p}}),$$

and in this case one shows that the image of this map is contained in U_p ordinary part $e(U_p)H^0(\mathrm{Sh}, \omega^{\mathbf{p}})$. Indeed, the Eichler-Shimura congruence relation shows that the U_p-operator restricted to the image of h satisfies the polynomial equation

$$U_p^2 - U_p \circ T_p + S_p = 0 \tag{1.0.3}$$

and since S_p is invertible, the constant term will be non-zero, hence this implies that the image is ordinary.

The second step is to show a doubling argument. We want to show that the given cohomological class produces two ordinary classes, that is, two U_p-eigenclasses in $e(U_p)H^0(\mathrm{Sh}, \omega^{\mathbf{p}})$, whose eigenvalues necessarily satisfy the above polynomial (1.0.3). In this case, one shows that the image of h is not a U_p-eigenclass. We see this by considering the partial Frobenius F operator, which is (essentially) a dual to U_p. We observe that if we did not have doubling, then we would also have an eigenvector for F, which is not possible.

After showing these two steps, one can prove the desired result by observing that local-global compatibility for ordinary Hilbert modular forms in regular weight implies that the Galois representation has a one dimensional unramified subspace on which Frobenius acts via the U_p-eigenvalue. If the two ordinary classes we produced had distinct eigenvalues (i.e. if the polynomial equation (1.0.3) has distinct roots) then we may easily conclude that the Galois representation is unramified. Even without the assumption of distinct eigenvalues, the argument of Dimitrov-Wiese [DW18] still allows us to get the desired conclusion, assuming we have shown doubling.

We can generalize the first step of the argument in the following way. Firstly, the work of Boxer-Pilloni [BP] on higher Hida theory for the Hilbert modular variety at hyperspecial level gives us a good understanding of the ordinary part of higher coherent cohomology in regular weight, and we modify their constructions to work for weight one. Let us briefly recall their constructions here, which we detail in §5. Hecke operators acting on integral coherent cohomology of the Hilbert modular variety have been constructed by Emerton-Reduzzi-Xiao [ERX17b] and Fakhruddin-Pilloni [FP21]. We have the T_p-operator acting via a normalized push-pull action on the coherent cohomology of Sh, and supported on the Hilbert modular variety at Iwahori level $\mathrm{Sh}_{\mathrm{I_p}}$. The mod p fiber of $\mathrm{Sh}_{\mathrm{I_p}}$ admits a stratification in terms of Kottwitz-Rapoport (KR) strata, and the maximal strata are indexed by subsets I of $\{1,\ldots,d\}$. Let X_I denote the closure of the maximal KR strata given by I. We have 2^d-possible U_p-operators, which we denote by $U_{p,I}$, each supported respectively on X_I. Moreover, we have that T_p is equal to $U_{p,I}$ when $\underline{\kappa}$ satisfies $\kappa_\tau \le 0$ for $\tau\in I$, $\kappa_\tau\ge 2$ for $\tau \in I^c$.

Boxer-Pilloni construct a geometric Jacquet-Langlands isomorphism which relates the $U_{p,I}$-ordinary parts of degree $j=\#I$ cohomology with the U_p^I-ordinary degree 0 cohomology of a quaternionic Shimura variety. Note that j is the only degree for which the non-Eisenstein $U_{p,I}$-ordinary part will be non-zero. They construct an isomorphism

$$e(U_{p,I})H^j(\mathrm{Sh}_{\overline{\mathbb{F}}_p},\omega^{\underline{\kappa}}) \simeq e(U_p^I)H^0(Z_{I,\overline{\mathbb{F}}_p},\omega^{\underline{\kappa}_I}) \tag{1.0.4}$$

where Z_I is a quaternionic Shimura variety, and $\underline{\kappa}_I$ is some automorphic weight, both determined explicitly by the set I. Here, the U_p^I-operator on Z_I is the standard one acting on quaternionic automorphic forms. As a consequence of this control theorem, Boxer-Pilloni also show local-global compatibility by constructing a Hida complex supported in the expected degrees over the Iwasawa algebra which interpolates the ordinary part of cohomology. We can hence also see that ordinary classes lift to characteristic zero, and hence the Galois representations associated with these classes contain a one-dimensional unramified subspace on which the Frobenius at p acts via the $U_{p,I}$-eigenvalue, exactly as in the description for degree zero.

We now describe how to make use of this construction in weight one. The first observation is that we can weight shift using partial Hasse invariants h_τ and partial theta operators θ_τ, to go from weight one into regular weight. More precisely, we show that for any non-Eisenstein cohomology class v in $H^j(\mathrm{Sh}_{\overline{\mathbb{F}}_p},\omega^{\mathbf{1}})$, there is some subset $S\subset\Sigma$ of places such that for all $\tau\notin S$, we have $h_\tau(v),\theta_\tau(v)\neq 0$, and for all $\tau\in S$, we have v lies in the image of h_τ,θ_τ. Moreover, we have $\#S=j$. This is achieved by using the description of coherent cohomology using the Cousin complex. This is the main result of §6.

From this set S, we can define $I=\sigma^{-1}(S)$. This will be the set I for which we want to show that v is ordinary for the $U_{p,I}$-operator. Following (1.0.4), we see that we want to shift the class into $H^0(Z_{I,\overline{\mathbb{F}}_p},\omega^{\underline{\kappa}_I})$. Such a shift is constructed via the maps

$$H^j(\mathrm{Sh}_{\overline{\mathbb{F}}_p},\omega^{\mathbf{1}}) \xrightarrow{i^*} H^j(Y_I,\omega^{\mathbf{1}}) \xleftarrow{j_!} H^0(Z_{I,\overline{\mathbb{F}}_p},\omega^{\underline{\kappa}_I}),$$

where Y_I is generalized Goren-Oort strata (as defined by Tian-Xiao [TX19]) of codim j associated with I. The map i^* will be the restriction map via the inclusion $i: Y_I\hookrightarrow \mathrm{Sh}_{\overline{\mathbb{F}}_p}$, while $j_!$ will be the pushforward via the map $j: Z_I\hookrightarrow Y_I$ induced by identifying Z_I with the closure of a central leaf in Y_I. The line bundle $\omega^{\underline{\kappa}_I}$ here will be the line bundle associated with regular weight $\underline{\kappa}$

such that

$$\kappa_\tau = \begin{cases} p & \text{if } \tau \notin I \\ 2-p & \text{if } \tau \in I. \end{cases}$$

We want to show that there exists some non-zero w such that $i_*(v) = j_!(w)$. Note that since the maps are equivariant for prime-to-p Hecke operators, v, w have the same system of Hecke eigenvalues and hence the same associated Galois representation. To see that such w exists, our first observation is that this would follow if we knew that v lay in the image of the map $k_! : H^0(Y_{I^c}, k^! \omega^{\mathbf{1}}) \to H^j(\mathrm{Sh}_{\overline{\mathbb{F}}_p}, \omega^{\mathbf{1}})$, where $I^c = \Sigma \backslash I$, since Y_I and Y_{I^c} intersect transversally, with intersection Z_I. We further see that this would follow if we knew that there was some cohomological correspondence C acting on $\mathrm{Sh}_{\overline{\mathbb{F}}_p}$ satisfying $p_1(C) = Y_{I^c}$, $p_2(C) = Y_I$ and such that the induced map T_C on $H^j(\mathrm{Sh}_{\overline{\mathbb{F}}_p}, \omega^{\mathbf{1}})$ has eigenvector v with non-zero eigenvalue. We show something like this is almost true after pre-and post composing T_{p^n} by some set of partial theta operators and Hasse invariants (depending on the set I), and showing that the support of the induced cohomological correspondence is only on those components satisfying $p_1(C) = Y_{I^c}$, $p_2(C) = Y_I$. This is Theorem 7.2.2, and is shown by studying components in the T_{p^n}, using a description of mod p irreducible components from [Lee21].

Since the combinatorics of the Hasse and theta operators from the set I can be somewhat complicated, we illustrate what is happening in the simplest case $d = 3$, and $H^1(\mathrm{Sh}_{\overline{\mathbb{F}}_p}, \omega^{\mathbf{1}})$. If we let $\Sigma = \{1, 2, 3\}$, the set S will be one of $\{1\}, \{2\}, \{3\}$. Let us suppose $S = \{2\}$, hence $I = \{1\}$. Then Z_I can be identified with the closure of the central leaf of $\mathrm{LT}_1 \times \mathrm{LT}_{23}$. Here $Y_I = V(h_2)$ is the vanishing locus of the partial Hasse invariant h_2, and $Y_{I^c} = V(h_1)$ is the vanishing locus of h_1. We observe then that the composition

$$T_{p,\{1\}} : H^1(\mathrm{Sh}_{\overline{\mathbb{F}}_p}, \omega^{\mathbf{1}-\boldsymbol{\theta_2}}) \xrightarrow{\theta_2} H^1(\mathrm{Sh}_{\overline{\mathbb{F}}_p}, \omega^{\mathbf{1}}) \xrightarrow{T_p} H^1(\mathrm{Sh}_{\overline{\mathbb{F}}_p}, \omega^{\mathbf{1}}) \xrightarrow{\theta_1 \circ h_3} H^1(\mathrm{Sh}_{\overline{\mathbb{F}}_p}, \omega^{\mathbf{1}+\theta_1+h_3})$$

is supported exactly on the strata $X_{\{1\}}$, where we abuse notation to denote the weight $\theta_i = (\dots, p, 1, \dots)$ to be the shift in weight induced by θ_i, i.e. $+1$ at i, $+p$ at $i-1$ and zero everywhere else, and similarly for $h_i = (\dots, p, -1, \dots)$. Hence, we have a commutative diagram

$$\begin{array}{ccc} H^1(\mathrm{Sh}_{\overline{\mathbb{F}}_p}, \omega^{\mathbf{1}-\boldsymbol{\theta_2}}) & \xrightarrow{T_{p,\{1\}}} & H^1(\mathrm{Sh}_{\overline{\mathbb{F}}_p}, \omega^{\mathbf{1}+\theta_1+h_3}) \\ \downarrow{\scriptstyle i^*} & & \uparrow{\scriptstyle k_!} \\ H^1(V(h_1), \omega^{\mathbf{1}-\boldsymbol{\theta_2}}) & \longrightarrow & H^0(V(h_2), k^! \omega^{\mathbf{1}+\theta_1+h_3}) \end{array}$$

Note that v is of the form $\theta_2(v')$ and we have $v'' = \theta_1 \circ h_3(v) \neq 0$ by assumption on I. Assuming $T_p(v) \neq 0$, we see that this means $T_{p,\{1\}}(v') \neq 0$, and thus $i^*(v) \neq 0$ and v'' lies in the image of $k_!$, the pushforward map from $V(h_2)$. Observing that θ_i is an isomorphism on $V(h_i)$, this gives us a class $w \in H^0(Z_I, k^! \omega^{1+\theta_1-\theta_2+h_3})$ which we want. Note that in this case the line bundle $\omega^{\underline{\kappa}_I}$ is exactly $k^! \omega^{1+\theta_1-\theta_2+h_3} = \omega^{1+\theta_1-\theta_2+h_3+h_2}$. In general we might not have $T_p(v) \neq 0$, but we will have $T_{p^n}(v) \neq 0$ for suficiently large n.

Finally, assuming such w exists, we show that the U_p^I operator acting on w satisfies the equation

$$(U_p^I)^2 - U_p^I \circ T_p + S_p^I = 0,$$

where for the composition $U_p^I \circ T_p$ we mean the image of U_p^I acting on a lift of $T_p(v)$ to $H^0(Z_{I,\overline{\mathbb{F}}_p}, \kappa_I)$, which we further show to be the same as a suitably defined 'T_p-operator' acting on w. The key technical input needed to prove this polynomial equation also comes from [Lee21] to understand how composition of various Hecke operators interacts with the geometric Jacquet-Langlands map of Boxer-Pilloni. More precisely, we show that $U_p^I \circ T_p$, as a cohomological correspondence, is supported on a Hecke correspondence which is the union of two components, namely a '$U_{p,I,V}$' and '$U_{p,I,F}$' component. When restricted to Z_I, these two components, over the μ-ordinary locus, agree with the cohomological correspondence of U_p^I and an associated 'Frobenius' F_p^I (dual to U_p^I) operator, respectively. From this analysis, the Eicher-Shimura relation for Z_I allows us to conclude the desired polynomial relation, and hence deduce that w is U_p^I-ordinary. This is also where the condition on the residual degree being odd is needed, as this description of the composition and the polynomial relation only holds when Z_I is the central leaf of a two-slope p-divisible group, ie. $\#I \neq \#I^c$.

Now, we need to show doubling in the ordinary Hecke algebra. As in the degree zero case, this amounts to showing that $U_p^I(w)$ is not a scalar multiple of w. The key observation we need here is that if $U_p^I(w)$ was a scalar multiple of w, then this would imply that w will be an eigenvalue of F. In particular, this implies that w gives us an F-ordinary class. Boxer-Pilloni [BP] show that, given the restriction on the cohomological degree if $\#I \neq \#I^c$, there are no nontrivial such classes. Once we have established doubling, essentially the same arguments of Dimitrov-Wiese [DW18] allow us to conclude that the Galois representation is unramified.

One can ask for a similar unramifiedness result for the mod p^n weight one Hecke algebra, not just individual mod p eigenclasses. Moreover, these methods carry over almost exactly to quaternionic Shimura varieties. We believe the techniques in this paper can be adapted to these situations, and we will pursue this in future work.

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2. Notation

(1) Let F be a totally real field. We fix a prime ideal $\mathfrak{p}$ of F above p, which we assume to be unramified, and denote by f its residual degree. We denote by O_F the ring of integers of F and the group of totally positive units by $O_F^{\times,+}$. Let ϖ be a uniformizer for $O_{F,\mathfrak{p}}$.

(2) Denote by $\mathfrak{d} := \mathfrak{d}_F$ the different ideal of $F/\mathbb{Q}$. Let $\mathfrak{C} := \{\mathfrak{c}_1, \dots, \mathfrak{c}_{h^+}\}$ be a fixed set of representatives for the elements of the narrow class group of F, chosen to be coprime to p. For a nonzero ideal $\mathfrak{a}$ of F, we write $N(\mathfrak{a})$ for its norm $\#(O_F/\mathfrak{a})$.

(3) Fix an isomorphism $\iota : \overline{\mathbb{Q}}_p \xrightarrow{\sim} \mathbb{C}$. We let Σ denote the set of real embeddings $\mathrm{Hom}(F, \mathbb{R})$, and denote by $\Sigma_\mathfrak{p}$ denote the subset of Σ whose composition with ι^{-1} induces $\mathfrak{p}$. We let τ denote an element of Σ, and we will use τ to also denote the induced p-adic embedding.

(4) Let $G = \mathrm{Res}_{F/\mathbb{Q}}\mathrm{GL}_2$, and denote by μ the cocharacter given by $\mathrm{diag}(1, 0)$.

(5) For some compact open subgroup $K \subset G(\mathbb{A}_f)$, we denote by Sh_K the integral model over $O_{F,\mathfrak{p}}$ of the Hilbert modular variety with level K. We assume that $K = K^pK_p$, where K_p is hyperspecial, and for notational simplicity we will often simply write Sh. We denote by $\mathrm{Sh}_{\mathrm{Iw}_\mathfrak{p}}$ the Hilbert modular variety with Iwahori level at $\mathfrak{p}$, and prime-to-p level K^p.
(6) For a set I, we denote by Y_I the generalized Goren-Oort strata associated with I.

3. Hilbert Modular Varieties

In this section, we will recall definitions and results of Tian-Xiao [ERX17a], [TX16] about the geometry of Hilbert modular varieties, at both hyperspecial and Iwahori levels. Note that the descriptions are simpler than what is stated in [ERX17a], as we are assuming that p is unramified, and hence we do not need to deal with Pappas-Rapoport splitting models.

3.1. **Definition.** Let S be a locally Noetherian O_F-scheme. A Hilbert–Blumenthal abelian S-scheme (HBAS) with real multiplication by O_F is the datum of an abelian S-scheme $\mathcal{A}$ of relative dimension d, together with a ring embedding $O_F \to \mathrm{End}_S\mathcal{A}$.

Let $\mathfrak{c} \in \mathfrak{C}$ be a fractional ideal of F coprime to $\mathfrak{p}$, with cone of positive elements $\mathfrak{c}^+$. A $\mathfrak{c}$-polarization on a HBAS over S is an S-isomorphism

$$\lambda : A \otimes_{O_F} \mathfrak{c} \xrightarrow{\sim} A^\vee$$

of HBASs under which the symmetric elements (resp. the polarizations) of $\mathrm{Hom}_{O_F}(A, A^\vee)$ correspond to the elements of $\mathfrak{c}$ (resp. $\mathfrak{c}^+$) in $\mathrm{Hom}_{O_F}(A, A \otimes_{O_F} \mathfrak{c})$.

Let $\hat{O}_F^{(p)}$ denote the direct product of completions of O_F at all finite places relatively prime to p. For a positive integer N relatively prime to p, a principal level N-structure on a HBAS $\mathcal{A}$ over S is an O_F-linear isomorphism of finite étale group schemes over S:

$$\alpha_N : (O_F/NO_F)^{\oplus 2} \xrightarrow{\sim} A[N].$$

This is the prime-to-p level structure attached to the open compact subgroup

$$K(N)^p := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\hat{O}_F^{(p)}) : a-1, b, c, d-1 \equiv 0 \mod N \right\}.$$

For a general K^p open compact subgroup of $\mathrm{GL}_2(\hat{O}_F^{(p)})$, choose a positive integer N relatively prime to p such that $K^p \subset K(N)^p$. Then a K^p-level structure on a HBAS $\mathcal{A}$ over S is a collection of, for each connected component S_i of S with a fixed geometric point $\bar{s}_i \in S_i$, a $\pi_1(S_i, \bar{s}_i)$-invariant $K^p/K(N)^p$-orbit of isomorphisms $\alpha_{N,\bar{s}_i}$. This does not depend on the choices of N and $\bar{s}_i$.

We let $K = K_pK^p$, where $K_p = \mathrm{GL}_2(O_{F,p})$. Denote by $\mathcal{M}_\mathfrak{c} := \mathcal{M}_{\mathfrak{c},K}$ the functor that assigns to any locally Noetherian O-scheme S the set of isomorphism classes of tuples $(\mathcal{A}, \lambda, \alpha)$, where

(1) $\mathcal{A}, \lambda$ is a $\mathfrak{c}$-polarized HBAS over S with real multiplication by O_F,
(2) α is a K^p-level structure, and
(3) $\pi_*\Omega_{\mathcal{A}/S}$ is a locally free $O_F \otimes_\mathbb{Z} O_S$-module of rank 1.

For sufficiently small K^p, this functor is representable by an O_F-scheme of finite type, which is moreover smooth. Let $\pi : \mathcal{A}_\mathfrak{c} \to \mathcal{M}_\mathfrak{c}$ denote the universal abelian scheme over $\mathcal{M}_\mathfrak{c}$. We have a natural direct sum decomposition

$$\omega := \pi_* \Omega_{\mathcal{A}_{\mathfrak{c}}/\mathcal{M}_{\mathfrak{c}}} = \bigoplus_{\tau : F \hookrightarrow \mathbb{R}} \omega_\tau$$

where each ω_τ is line bundle on $\mathcal{M}_{\mathfrak{c}}$.

Note that $\mathcal{M}_{\mathfrak{c}}$ has an action of $O_F^{\times,+}$, namely by sending $(\mathcal{A}, \lambda, \alpha) \to (\mathcal{A}, u\lambda, \alpha)$ for any $u \in O_F^{\times,+}$. This action is trivial on the subgroup $(K \cap O_F^\times)^2$. From now on, we will also K^p is sufficiently small so that the action of $\Delta := O_F^{\times,+}/(K \cap O_F^\times)^2$ is free on geometric points.

Denote

$$\mathrm{Sh}_{\mathfrak{c}} := \mathcal{M}_{\mathfrak{c}}/\Delta,$$

and the Hilbert modular variety is defined as the disjoint union

$$\mathrm{Sh} := \coprod_{\mathfrak{c}} \mathrm{Sh}_{\mathfrak{c}}.$$

Each line bundle ω_τ descends to a line bundle on Sh. Moreover, we have another line bundle on $\mathcal{M}_{\mathfrak{c}}$, ϵ_τ, defined as follows.

Consider $H^1_{dR}(\mathcal{A}_{\mathfrak{c}}/\mathcal{M}_{\mathfrak{c}}) = R^1\pi_*\Omega_{\mathcal{A}_{\mathfrak{c}}/\mathcal{M}_{\mathfrak{c}}}$, which is locally free of rank 2 over $O_F \otimes O_{\mathcal{M}_{\mathfrak{c}}}$. Then, the exterior product

$$\epsilon := \bigwedge^2_{O_F \otimes O_{\mathcal{M}_{\mathfrak{c}}}} H^1_{dR}(\mathcal{A}_{\mathfrak{c}}/\mathcal{M}_{\mathfrak{c}})$$

decomposes into a direct sum of line bundles ϵ_τ, each of which is trivial, but which has a non-trivial O_F-action. ϵ_τ also descends to a trivial line bundle on Sh.

3.1.1. We will choose a toroidal compactification of $\mathcal{M}_{\mathfrak{c}}$, denoted by $\mathcal{M}_{\mathfrak{c}}^{\mathrm{tor}}$, associated to some fixed smooth rational polyhedral admissible cone decomposition for each cusp of $\mathcal{M}_{\mathfrak{c}}$. (This does not depend on $\mathfrak{c}$.) The action of Δ extends to $\mathcal{M}_{\mathfrak{c}}^{\mathrm{tor}}$.

Like before, we let $\mathrm{Sh}^{\mathrm{tor}}$ denote $\coprod_{\mathfrak{c}} \mathcal{M}_{\mathfrak{c}}^{\mathrm{tor}}/\Delta$, the union of quotients by the action of the group Δ. The boundary divisor $D = \mathrm{Sh}^{\mathrm{tor}} - \mathrm{Sh}$ is a divisor with simple normal crossings.

3.1.2. A *paritious* weight $\underline{\kappa}$ is a tuple $((\kappa_\tau)_{\tau\in\Sigma}, w) \in \mathbb{Z}^\Sigma \times \mathbb{Z}$ such that $\kappa_\tau \equiv w \pmod 2$ for every $\tau \in \Sigma$. For an integer n, we write $\mathbf{n} = (n, \dots, n)$.

For any paritious weight $\underline{\kappa}$, we have a line bundle

$$\omega^{\underline{\kappa}} := \bigotimes_{\tau\in\Sigma} \omega_\tau^{\otimes\kappa_\tau} \otimes \epsilon_\tau^{\otimes\frac{w-k_\tau}{2}}.$$

The coherent cohomology groups of weight κ are the groups

$$H^\bullet(\mathrm{Sh}^{\mathrm{tor}}, \omega^{\underline{\kappa}}), \text{ and } H^\bullet(\mathrm{Sh}^{\mathrm{tor}}, \omega^{\underline{\kappa}}(-D)).$$

Note that while the compactification $\mathrm{Sh}^{\mathrm{tor}}$ clearly depends on the choice of the smooth rational polyhedral cone decompositions, the cohomology groups defined are in fact independent of this choice. Indeed given some refinement rational polyhedral cone decomposition, giving rise to some other toroidal compactification $\tilde{\mathrm{Sh}}^{\mathrm{tor}}$, we get a proper map $\tilde{\mathrm{Sh}}^{\mathrm{tor}} \to \mathrm{Sh}^{\mathrm{tor}}$, and the derived pushforward of the associated line bundles $\tilde{\omega}^{\underline{\kappa}}$ and $\tilde{\omega}^{\underline{\kappa}}(-D)$ along this proper projection is simply the line bundles $\omega^{\underline{\kappa}}$ and $\omega^{\underline{\kappa}}(-D)$, hence the cohomology groups are isomorphic [Lan13, Lemma 7.1.1.4].

3.2. **Ekedahl-Oort strata.** We recall some definitions from [TX16] about the Ekedahl-Oort stratification on the mod p Hilbert modular variety, and the structure as an iterated projective space bundle over some quaternionic Shimura variety. From now on, we will denote by $\mathcal{S}$ the special fiber $\mathrm{Sh}_{\overline{\mathbb{F}}_p}$

3.2.1. The Verschiebung map on the universal abelian variety $\mathcal{A}_{\mathfrak{c}}/\mathcal{M}_{\mathfrak{c},\overline{\mathbb{F}}_p}$ induces a map on line bundles

$$\omega \to \omega^{(p)},$$

which, under the decomposition $\omega = \oplus \omega_\tau$ using the O_F-action, induces for all $\tau \in \Sigma$ homomorphisms

$$h_\tau : \omega_\tau \to \omega_{\sigma^{-1}\tau}^{\otimes p}$$

Here σ is the Frobenius map on Σ, and acts as a cyclic permutation on each $\Sigma_{\mathfrak{p}}$. This map then defines a partial Hasse invariant $h_\tau \in H^0(\mathcal{M}_{\mathfrak{c},\overline{\mathbb{F}}_p}\omega_{\sigma^{-1}\tau}^{\otimes p} \otimes \omega_\tau^{\otimes -1})$. Thus, for every τ, we have a closed subscheme of $\mathcal{M}_{\mathfrak{c},\overline{\mathbb{F}}_p}$ which corresponds to the vanishing locus of h_τ. Taking unions over all $\mathfrak{c}$, then taking quotients, we obtain a closed subset $\mathcal{S}_\tau \subset \mathrm{Sh}_{\overline{\mathbb{F}}_p}$. Then Goren-Oort [GO00] prove that these closed subschemes are actually smooth, and for any $\tau \neq \tau'$, $\mathcal{S}_\tau$ and $\mathcal{S}'_\tau$ intersect transversally. Indeed, for any $I \subset \Sigma$, we can consider

$$\mathcal{S}_I := \cap_{\tau \in I} \mathcal{S}_\tau,$$

which is a smooth closed subscheme in $\mathcal{S}$ of dimension $d - \#I$. Moreover, Goren-Oort [GO00] also show that these strata are closely related to another stratification on the Shimura variety, namely the Ekedahl-Oort stratification. Indeed, we see that Goren-Oort strata are simply closures of Ekedahl-Oort strata.

Observe from this description that the normal bundle of the regular immersion $\mathcal{S}_I/\mathcal{S}$ is exactly

$$\mathcal{N}_{\mathcal{S}_I/\mathcal{S}} = \bigoplus_{\tau \in I} \mathcal{L}_\tau \mid_{\mathcal{S}_I},$$

where $\mathcal{L}_\tau = \omega_{\sigma^{-1}\tau}^{\otimes p} \otimes \omega_\tau^{\otimes -1}$.

Note that the partial Hasse invariants extend to the toroidal compactification $\mathcal{S}^{\mathrm{tor}}$, to define closed subschemes of $\mathcal{S}^{\mathrm{tor}}$. However, we see that the vanishing locus of h_τ in $\mathcal{S}^{\mathrm{tor}}$ is still just $\mathcal{S}_\tau$, and entirely contained in the open Shimura variety.

Moreover, $\mathcal{S}_J$ has the structure as some kind of iterated projective space bundle over a quaternionic Shimura variety, the structure of which we now recall.

Theorem 3.2.2 ([TX16, Theorem 5.8])**.** *There is some set I_J such that $\mathcal{S}_J$ is a $(\mathbb{P}^1)^{\#I_J}$-bundle over a quaternionic Shimura variety Sh_{B_J}.*

3.3. **Generalized Goren-Oort strata.** We recall in this section the descriptions of generalized Goren-Oort strata from Tian-Xiao [TX19], and relate to each set $I \subset \Sigma_{\mathfrak{p}}$ a generalized Goren-Oort strata of codimension $\#I$.

Let $I_1^+ := \{\tau \in I : \sigma^{-1}\tau \notin I\}$. In this way, we see that we can partition the set I into sets of consecutive elements in I, and elements in I^+ are the first element of such sets. By construction I^+ cannot contain any consecutive elements.

Following Theorem 3.2.2, we see that the set $\mathcal{S}_{I_1^+}$ is a $\mathbb{P}^{\#I_1^+}$-bundle over a quaternionic Shimura variety Sh_{B_1}, where here B_1 is the quaternion algebra ramified at $\{\tau, \sigma^{-1}\tau : \tau \in I_1^+\}$. Consider now the set I_2^+, which is constructed as for I_1^+ from the set of unramified places of B_1: we partition the remaining elements of I into consecutive elements, and take the first element of each subset. The Goren-Oort strata Sh_{B_1,I_2^+} of Sh_{B_1} given by the vanishing of the partial Hasse invariants for I_2^+. We now observe that Sh_{B_1,I_2^+} is also some projective space bundle over a smaller quaternionic Shimura variety.

Thus, we see that we can iterate this construction for $n = 2, 3\dots$ until we get the first i such that $I_i^+ = \emptyset$. Then, we define Y_I, the generalized Goren-Oort strata attached to I to be the preimage in $\mathcal{S}$ under all these projective bundle maps of $\mathrm{Sh}_{B_{i-1}}$.

3.4. **Iwahori level.** For every fractional ideal $\mathfrak{c} \in \mathfrak{C}$ we choose once and for all a positive isomorphism $\theta_{\mathfrak{c}} : \mathfrak{c}\mathfrak{p} \simeq \mathfrak{c}'$ of $\mathfrak{c}\mathfrak{p}$ with a (uniquely determined) fractional ideal $\mathfrak{c}' \in \mathfrak{C}$.

Observe that the group $\mathrm{Res}_{F/\mathbb{Q}}\mathrm{GL}_2$ decomposes as $\prod_{\mathfrak{p}|p} \mathrm{Res}_{F_\mathfrak{p}/\mathbb{Q}_p}\mathrm{GL}_2$. Let $I_\mathfrak{p} \subset \mathrm{GL}_2(F_\mathfrak{p})$ denote the Iwahori subgroup. We consider now the situation where the level at p is the parahoric subgroup of the form $\mathrm{Iw}_\mathfrak{p} = I_\mathfrak{p} \prod_{\mathfrak{p}' \neq \mathfrak{p}} \mathrm{GL}_2(O_{F,\mathfrak{p}'})$. For a fixed $\mathfrak{c} \in \mathfrak{C}$ we define the models with $\mathrm{Iw}_\mathfrak{p}$-level structure at p, denoted by $\mathcal{M}_{\mathfrak{c},\theta_c}(\mathfrak{p})$ as the following functor, which takes any locally noetherian O_F-scheme S to the set of isomorphism classes of tuples $((\mathcal{A}, \lambda, \alpha); (\mathcal{A}', \lambda', \alpha'); \varphi, \psi)$ where

1. $(\mathcal{A}, \lambda, \alpha) \in \mathcal{M}_\mathfrak{c}(S)$, and $(\mathcal{A}', \lambda', \alpha') \in \mathcal{M}_{\mathfrak{c}'}(S)$
2. $\varphi : \mathcal{A} \to \mathcal{A}'$ and $\psi : \mathcal{A}' \to \mathcal{A} \otimes \mathfrak{c}(\mathfrak{c}')^{-1}$ are O_F -equivariant S-isogenies satisfying:
   (a) both φ and ψ have degree p^f,
   (b) The compositions $\psi\circ\varphi$ and $(\varphi\otimes\mathfrak{c}(\mathfrak{c}')^{-1})\circ\psi$ are the natural isogenies $\mathcal{A} \to \mathcal{A}\otimes\mathfrak{c}(\mathfrak{c}')^{-1}$ and $\mathcal{A}' \to \mathcal{A}' \otimes \mathfrak{c}(\mathfrak{c}')^{-1}$ induced by $O_F \subset \mathfrak{p}^{-1} \xrightarrow[\sim]{\theta_\mathfrak{c}} \mathfrak{c}(\mathfrak{c}')^{-1}$,
   (c) φ, ψ are compatible with the polarizations, i.e., $\varphi \circ \lambda \circ \varphi^\vee = \lambda'$, where the map λ' is $(\mathcal{A}')^\vee \to \mathcal{A}' \otimes \mathfrak{c}$ induced by composing λ with the map $\mathfrak{c}' \xrightarrow[\sim]{\theta_\mathfrak{c}^{-1}} \mathfrak{c}\mathfrak{p} \subset \mathfrak{c}$,
   (d) both φ and ψ are compatible with the level structures, i.e., $\varphi \circ \alpha = \alpha'$ and $\varphi \circ \alpha' = \alpha \otimes \mathfrak{c}(\mathfrak{c}')^{-1}$,

$\mathcal{M}_{\mathfrak{c},\theta_c}(\mathfrak{p})$ is representable by an $O_{F,\mathfrak{p}}$ scheme of finite type. Moreover, we have natural projection maps

$$p_1 : \mathcal{M}_{\mathfrak{c},\theta_c}(\mathfrak{p}) \to \mathcal{M}_\mathfrak{c}$$
$$p_2 : \mathcal{M}_{\mathfrak{c},\theta_c}(\mathfrak{p}) \to \mathcal{M}_{\mathfrak{c}'},$$

given by taking the tuple $((\mathcal{A}, \lambda, \alpha); (\mathcal{A}', \lambda', \alpha'); \varphi, \psi)$ to $(\mathcal{A}, \lambda, \alpha)$ and $(\mathcal{A}', \lambda', \alpha')$ respectively, which are moreover proper morphisms.

The group $O_F^{\times,+}/(K \cap O_F^{\times,+})^2$ acts freely on $\mathcal{M}_{\mathfrak{c},\theta_c}(\mathfrak{p})$, via its simultaneous action on both $\mathcal{A}$ and $\mathcal{A}'$. We let $\mathrm{Sh}_\mathfrak{c}(\mathfrak{p})$ be the corresponding quotient. Similar to the case of hyperspecial level, we define

$$\mathrm{Sh}_{\mathrm{Iw}_\mathfrak{p}} := \coprod_{\mathfrak{c}\in\mathfrak{C}} \mathrm{Sh}_\mathfrak{c}(\mathfrak{p}).$$

For each $\mathfrak{c}$, we have universal isogenies $\varphi_\mathfrak{c} : \mathcal{A}_{1,\mathfrak{c}} \to \mathcal{A}_{2,\mathfrak{c}}$ and $\psi_\mathfrak{c} : \mathcal{A}_{2,\mathfrak{c}} \to \mathcal{A}_{1,\mathfrak{c}}$ over the universal abelian varieties $\pi_1 : \mathcal{A}_{1,\mathfrak{c}} \to \mathcal{M}_{c,\theta_\mathfrak{c}}(\mathfrak{p})$, $\pi_2 : \mathcal{A}_{2,\mathfrak{c}} \to \mathcal{M}_{c,\theta_\mathfrak{c}}(\mathfrak{p})$.

3.4.1. There is a toroidal compactification of $\mathrm{Sh}_{\mathrm{Iw}_\mathfrak{p}}$, denoted by $\mathrm{Sh}^{\mathrm{tor}}_{\mathrm{Iw}_\mathfrak{p}}$ constructed from the toroidal compactification of $\mathcal{M}_{\mathfrak{c},\theta_\mathfrak{c}}$ as constructed by Lan [Lan13]. Again, as for hyperspecial level, we may quotient by Δ, and take the union over all $\mathfrak{c}$.

The definition of the projection maps p_1, p_2 extends to $\mathrm{Sh}_{\mathrm{Iw}_\mathfrak{p}}$, and the toroidal compactification $\mathrm{Sh}^{\mathrm{tor}}_{\mathrm{Iw}_\mathfrak{p}}$. From now on, we will denote the mod p fiber of $\mathrm{Sh}_{\mathrm{Iw}_\mathfrak{p}}$ by $T_\mathfrak{p}$, with toroidal compactification

$T_{\mathfrak{p}}^{\text{tor}}$, and thus we have an algebraic correspondence

$$\begin{array}{ccc} & T_{\mathfrak{p}}^{(\text{tor})} & \\ {}^{p_2}\swarrow & & \searrow^{p_1} \\ \mathcal{S}^{(\text{tor})} & & \mathcal{S}^{(\text{tor})}. \end{array}$$

3.4.2. There is a Kottwitz-Rapoport (KR) stratification on $T_{\mathfrak{p}}^{\text{tor}}$, indexed by subsets $I \subset \Sigma_{\mathfrak{p}}$, defined as follows for all $\mathfrak{c}$.

We can consider the map on vector bundles

$$\varphi : p_2^*\omega = \omega_{\mathcal{A}_2} \to \omega_{\mathcal{A}_1} = p_1^*\omega$$

induced by the universal isogeny $\varphi_{\mathfrak{c}}$ over $\mathcal{M}_{c,\theta_{\mathfrak{c}}}(\mathfrak{p})$. Taking quotients by Δ everywhere gives us a map

$$p_2^*\omega \to p_1^*\omega$$

over $\text{Sh}_{\mathfrak{c}}(\mathfrak{p})$. Moreover, decomposition according to the O_F-action gives us for each $\tau \in \Sigma_{\mathfrak{p}}$ a map of line bundles

$$\phi_\tau : p_2^*\omega_\tau \to p_1^*\omega_\tau.$$

We can similarly define such a map from $\psi_{\mathfrak{c}}$, and take unions over all $\mathfrak{c}$.

For all subsets $I \subset \Sigma_{\mathfrak{p}}$, we can consider the vanishing locus in $T_{\mathfrak{p}}$ of the product

$$\prod_{\tau \in I} \varphi_\tau \prod_{\tau \notin I} \psi_\tau,$$

and further the closure of this subset in $T_{\mathfrak{p}}^{\text{tor}}$. We denote this closed subscheme by X_I. Note that X_I is equidimensional of dimension d.

This closed subscheme can also be defined in terms of the local model for $\text{Sh}_{\text{Iw}_{\mathfrak{p}}}$, which we now recall.

3.4.3. Let $V_0 = \mathbb{Z}^2$, with symplectic pairing ψ given by the matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Consider the chain of modules

$$V_\bullet : V_0 \to V_1 \to V_0,$$

where V_1 is also a free rank 2 $\mathbb{Z}$-module, and the map from V_i to V_{i+1} is given in the canonical basis by the map $e_j \mapsto e_j$ if $j \neq i+1$ and $e_{i+1} \mapsto pe_{i+1}$. (We take $V_2 = V_0$)

For every subset $\emptyset \neq J \subset \{0, 1\}$, there is a local model M_J which is the projective scheme over $\mathbb{Z}$ which represents the functor which takes any $\mathbb{Z}$-algebra R to the set of isomorphism classes of diagrams

$$\begin{array}{ccccccc} V_{i_0} \otimes_{\mathbb{Z}} R & \longrightarrow & V_{i_1} \otimes_{\mathbb{Z}} R & \longrightarrow & \dots & \longrightarrow & V_{i_m} \otimes_{\mathbb{Z}} R \\ \uparrow & & \uparrow & & & & \uparrow \\ F_{i_0} & \longrightarrow & F_{i_1} & \longrightarrow & \dots & \longrightarrow & F_{i_m} \end{array}$$

where $i_0 < i_1 \cdots < i_m$ are such that $\{i_0, \dots, i_m\} = J \cup \{2 - i | i \in J\}$, and the modules F_{i_j}, for $0 \leq j \leq m$, are rank 1 locally direct factors of $V_{i_j} \otimes_{\mathbb{Z}} R$, which are self dual with respect to the perfect pairing induced by ψ (i.e. $F_{2-i}^\perp = F_i$ for all $i \in J$). When $J' \subset J$, there is an obvious map $M_J \to M_{J'}$ given by forgetting the modules F_j for $j \in J \backslash J'$. Moreover, there is a canonical identification of $M_{\{0\}} \simeq M_{\{1\}}$ given by taking F_1 to be F_0, via the tautological identification of V_0 and V_1.

3.4.4. The special fiber of the local model embeds into the affine flag variety (affine Grassmannian), which is defined as the following. Let $\mathcal{V}_0 = \mathbb{F}_p[[t]]^2$, with symplectic pairing given also by the matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Similarly to above, we now consider the chain:

$$\mathcal{V}_\bullet : \mathcal{V}_0 \to \mathcal{V}_1 \to \mathcal{V}_0$$

where $\mathcal{V}_1$ is also a free rank two $\mathbb{F}_p[[t]]$ module, and the map from $\mathcal{V}_i$ to $\mathcal{V}_{i+1}$ is given in the canonical basis by $e_j \mapsto e_j$ if $j \neq i+1$ and $e_{i+1} \mapsto pe_{i+1}$. (We take $\mathcal{V}_2 = \mathcal{V}_0$). For any subset $\emptyset \neq I \subseteq \{0,1\}$, we let $\mathcal{V}_\bullet^J$ be the subchain of $\mathcal{V}_\bullet$ where we keep only the modules indexed by elements i such that either i or $2-i$ is in J. Let LG denote the loop group of GL_2 over $\mathbb{F}_p$. The group LG acts on the chain $\mathcal{V}_\bullet$ via its action on $\mathcal{V}_0$ (as a lattice in $\mathcal{V}_0 \otimes_{\mathbb{F}_p[[t]]} \mathbb{F}_p((t))$), and for each subset J as above, we let $\mathcal{P}_J$ be the associated parahoric subgroup. The affine flag variety is $\mathcal{F}_J := LG/\mathcal{P}_J$.

We can identify the special fiber of the local model $\overline{M}_J$ with a finite union of $\mathcal{P}_J$ orbits in $\mathcal{F}_J$. Indeed, we have a map $\mathcal{M}_J \to \mathcal{F}_J$. Indeed, for any $\mathbb{F}_p$-algebra R, an R-point of $\overline{M}_J$ gives, for every i_j, a rank 1 locally direct factor F_{i_j} of $V_{i_j} \otimes_{\mathbb{Z}} R$. This uniquely determines an $R[[t]]$-submodule $\mathcal{F}_{i_j}$ of $\mathcal{V}_{i_j} \otimes_{\mathbb{F}_p} R$ which is locally free of rank 2, and containing $t\mathcal{V}_{i_j} \otimes_{\mathbb{F}_p} R$, such that the fiber of over the closed point $\{t=0\}$ is F_{i_j}. The chain $\mathcal{F}_\bullet$ determines an R-point of $\mathcal{F}_J$.

The main result about local models is the following theorem:

Theorem 3.4.5. *Let $\tilde{\mathcal{P}}$ (resp. $\tilde{\mathcal{P}}_{\mathrm{Iw_p}}$) denote the universal G^c-torsors over* Sh *(resp.* $\mathrm{Sh}_{\mathrm{Iw_p}}$*. We have a commutative diagram*

(3.4.6)
$$\begin{array}{ccccc}
 & & \tilde{\mathcal{P}}_{\mathrm{Iw_p}} & \xrightarrow{q_{\mathrm{Iw_p}}} & \prod_{\tau\in\Sigma_p} M_{\{0,1\}} \prod_{\tau\notin\Sigma_p} M_{\{0\}} \\
 & \swarrow^{p_{\mathrm{Iw_p}}} & \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle \pi_1} \\
\mathrm{Sh}_{\mathrm{Iw_p}} & & \tilde{\mathcal{P}} & \xrightarrow{q} & \prod_\tau M_{\{0\}}. \\
\downarrow & \swarrow^{p} & & & \\
\mathrm{Sh} & & & &
\end{array}$$

where the left arrows are both G^c-torsors, and the right arrows are G^c-equivariant. Moreover, these maps extend to the toroidal compactification.

3.4.7. Let $\tilde{W}$ be the extended affine Weyl group of GL_2. Concretely, this is the semi-direct product of the finite Weyl group W and the cocharacter group $X_*(T)$, where T is the diagonal torus in GL_2. It contains as a subgroup the affine Weyl group of GL_2 , which is a Coxeter group with simple reflection s_1 and one affine reflection s_0. For each subset J as above, we have a finite subgroup W_J of $\tilde{W}$ generated by the simple reflections s_i, $i \in J$. The $\mathcal{P}_J$ orbits in $\mathcal{F}_J$ are parametrised by the double cosets $W_J \backslash \tilde{W} / W_J$. The orbits included in $\overline{M}_J$ are parametrized by the finite subset $\mathrm{Adm}_J(\mu)$ of $W_J \backslash \tilde{W} / W_J$ of μ-admissible elements as defined in [HR17, 2.2].

We have two possibilities for I: $I = \{0\}$, or $I = \{0,1\}$. In the first case, the parahoric subgroup $\mathcal{P}_I$ is hyperspecial. The special fiber $\overline{M}_{\{0\}}$ is isomorphic to $\mathbb{P}^1$, hence smooth and irreducible. Moreover, the μ-admissible set is a single element.

We consider now the case of $I = \{0, 1\}$. In this case, we have the following well known result:

Proposition 3.4.8. *The special fiber* $\overline{M}_{\{0,1\}}$ *consists of two irreducible components, which are both isomorphic to* $\mathbb{P}^1$*, intersecting transversally at a point. Each of these irreducible components correspond to an open strata for the KR stratification, indexed by the integer* $0 \leq s \leq 1$*. For each such* s*, a representative of the* s*-stratum is given by taking*

$$F_0(s) = F_1(s) = \langle e_{s+1} \rangle,$$

corresponding to the element $\mathrm{diag}(t\mathrm{Id}_s, \mathrm{Id}_{1-s}, t\mathrm{Id}_{1-s}, \mathrm{Id}_s) \in LG$.

Note also that the open strata for $s = 1$ corresponds to $U_{\{0,1\},\mu}$, while for $s = 0$ it corresponds to $U_{\{0,1\},w\mu}$ where w is the non-trivial element of W.

For all $I \subset \Sigma_{\mathfrak{p}}$, we may thus also define the open Kottwitz-Rapoport strata $X_I^\circ \subset T_{\mathfrak{p}}^{\mathrm{tor}}$ to be the preimage, under the maps p, q of the local model diagram, of the product of the orbits

$$\prod_{\tau \in I} U_{\{0,1\},\mu} \prod_{\tau \in \Sigma_{\mathfrak{p}} - I} U_{\{0,1\},w\mu} \prod_{\tau \notin \Sigma_{\mathfrak{p}}} M_{\{0\}} \subset \prod_{\tau \in \Sigma_{\mathfrak{p}}} M_{\{0,1\}} \prod_{\tau \notin \Sigma_{\mathfrak{p}}} M_{\{0\}}.$$

The closures of X_I° is the closed subscheme X_I previously defined.

The local model diagram immediately gives us the following result.

Proposition 3.4.9. $T_{\mathfrak{p}}^{\mathrm{tor}}$ *is a local complete intersection, and the closures of Kottwitz-Rapoport strata are smooth.*

In particular, we may study individual X_I in $T_{\mathfrak{p}}^{\mathrm{tor}}$ separately, and we have for all I algebraic correspondences

$$\begin{array}{ccc} & X_I & \\ {\scriptstyle p_{2,I}} \swarrow & & \searrow {\scriptstyle p_{1,I}} \\ \mathcal{S}^{\mathrm{tor}} & & \mathcal{S}^{\mathrm{tor}}. \end{array}$$

The local model diagram (or rather the commutativity of (3.4.6)), which allows us to understand the projection maps p_1, p_2 in terms of maps of local models, gives us the following result

Proposition 3.4.10. *The projection maps* p_1, p_2 *for* $T_{\mathfrak{p}}^{\mathrm{tor}}$ *and* $p_{1,I}, p_{2,I}$ *for* $I \subset \Sigma_{\mathfrak{p}}$ *are lci morphisms.*

In the following section, we will define cohomological correspondences supported on $T_{\mathfrak{p}}^{\mathrm{tor}}$ and X_I.

4. Cohomological Correspondences

In this section, we recall the construction in [FP21, §2] of Hecke operators acting on the cohomology of coherent automorphic bundles. We fix for the entire section a smooth base scheme S, and assume that all schemes (unless otherwise stated) are S-schemes. All functors $\otimes, *, !$ are derived functors in the derived ∞-category $D_{qcoh}(X)$ of quasi-coherent sheaves on some S-scheme X.

4.1. Preliminaries.

4.1.1. Recall that for any morphism of schemes $f : X \to Y$, f is called a local complete intersection (lci) morphism if locally on X we can factorize f as the composition

$$f : X \xrightarrow{i} Z \xrightarrow{f'} Y,$$

where i is a regular embedding and f' is a smooth morphism. We also recall that if f is lci, then the relative cotangent complex $\mathbb{L}_{X/Y}$ is a perfect complex, with amplitude $[-1, 0]$. Moreover, if f is proper,then we know that the dualizing sheaf $f^! O_Y$ is equal to $\det \mathbb{L}_{X/Y}$.

4.1.2. We now construct fundamental classes for proper lci morphisms of virtual codimension 0. These are global sections of the dualizing sheaf $f^! O_Y$. Since $\mathbb{L}_{X/Y}$ is perfect, we can choose a presentation as a complex of locally free sheaves. In fact, locally on X we may further assume that the $\mathbb{L}_{X/Y}$ is concentrated only in degrees -1 and 0, and thus locally of the form

$$F^\bullet = [F_{-1} \xrightarrow{h} F_0].$$

Taking determinants, we have (locally) a map

$$\det F_{-1} \xrightarrow{\det(h)} \det F_0$$

which glues to a global section in $H^0(X, f^! O_Y)$, since the section on each local neighbourhood is independent of the choice of resolution $F^\bullet$, as shown in [Sta23, Tag 0FJI]. Now, note that $f^* O_Y$ is a complex concentrated in negative degrees, and we have the truncation map

$$f^* O_Y \to \tau^{\geq 0} f^* O_Y \simeq O_X.$$

We thus define the fundamental class as the composition

$$\Theta_{X/Y} : f^* O_Y \to O_X \to \det \mathbb{L}_{X/Y} = f^! O_Y$$

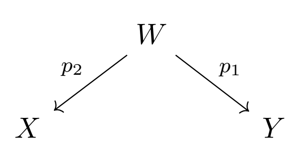
in $D_{qc}(O_X)$. Observe that any map

$$f^* O_Y \to f^! O_Y$$

in $D_{qc}(X)$ must factor through the truncation map $f^* O_Y \to \tau^{\geq 0} f^* O_Y$, because f^* is left t-exact, while $f^!$ is right t-exact. Thus $\Theta_{X/Y}$ uniquely determines (and is determined by) a map $O_X \to f^! O_Y$, i.e. a global section of the line bundle $f^! O_Y$.

Moreover, note that if we also impose the additional assumptions on $f : X \to Y$ as in [FP21, Prop 2.6], then what we have constructed agrees with the definition in [FP21, Prop 2.6]. Indeed since $\Theta_{X/Y}$ is determined by its restriction to some open subset $U \subset X$, and by shrinking U sufficiently, we may assume that there exists some $V \subset Y$ such that $U = f^{-1}(V)$, and both U, V are smooth over S. Then both $F^\bullet|_U$ and $[\Omega_{U/S} \xrightarrow{df} \Omega_{V/S}]$ are resolutions of $\mathbb{L}_{U/V}$, and hence, again by [Sta23, Tag 0FJI], the determinant is independent of the choice of resolution, so the element of $H^0(U, f^! O_V)$ defined is the same.

4.1.3. Now, suppose we have an algebraic correspondence, i.e. we have schemes X, Y, W and a diagram



such that the maps p_1, p_2 are proper (but not necessarily finite). Let $\mathcal{F} \in D^-_{qcoh}(O_X)$ and $\mathcal{G} \in D^+_{qcoh}(O_Y)$.

Definition 4.1.4. A cohomological correspondence from $(X, \mathcal{F})$ to $(Y, \mathcal{G})$ is the data of an algebraic correspondence W over X and Y and a map in $D_{qcoh}(O_W)$

$$T_W : p_2^*(\mathcal{F}) \to p_1^!(\mathcal{G}).$$

Equivalently, by adjunction, this is map $T_W : p_{1*}p_2^*(\mathcal{F}) \to \mathcal{G}$.

Observe that a cohomological correspondence induces a map from $R\Gamma(X, \mathcal{F}) \to R\Gamma(Y, \mathcal{G})$, defined by composing the maps

$$R\Gamma(X, \mathcal{F}) \xrightarrow{p_2^*} R\Gamma(W, p_2^*\mathcal{F}) \xrightarrow{T_W} R\Gamma(W, p_1^!\mathcal{G}) \xrightarrow{p_{1,!}} R\Gamma(Y, \mathcal{G})$$

4.1.5. Given a commutative diagram

$$\begin{array}{ccccc} X_1 & \xleftarrow{q_2} & W_1 & \xrightarrow{q_1} & Y_1 \\ \downarrow{\scriptstyle f_X} & & \downarrow{\scriptstyle f_W} & & \downarrow{\scriptstyle f_Y} \\ X_2 & \xleftarrow{p_2} & W_2 & \xrightarrow{p_1} & Y_2 \end{array} \tag{4.1.6}$$

where all the vertical maps are proper, we can decompose the left square as

$$\begin{array}{cccc} W_1 & & \overset{q_2}{\longrightarrow} & X_1 \\ & \searrow^{a} & & \\ & X_1 \times^{\mathbb{L}}_{X_2} W_2 & \xrightarrow{a_2} & X_1 \\ {\scriptstyle q_1}\searrow & \downarrow{\scriptstyle a_1} & & \downarrow \\ & W_2 & \longrightarrow & X_2 \end{array}$$

where we note that since f_W is proper, so is a. Thus, applying base change along the Cartesian square and adjunction for a^* gives us a map

$$p_2^*(f_{X*}\mathcal{F}) \to a_{1*}a_2^*\mathcal{F} \to a_{1*}a_*a^*a_2^*\mathcal{F} = f_{W*}q_2^*\mathcal{F}.$$

The same construction applied to the right square (where here we now apply adjunction for $a^!$ instead) gives us a map

$$f_{W*}q_1^!\mathcal{G} \to p_{1*}f_Y^!\mathcal{G}.$$

Definition 4.1.7. Given a commutative diagram as in (4.1.9), the pushforward of the cohomological correspondence $T_{W_1} : q_2^*\mathcal{F} \to q_1^!\mathcal{G}$ along f_W to W_2 is defined as the composition

$$T_{W_2} = f_*T_{W_1} := p_2^*(f_{X*}\mathcal{F}) \to f_{W*}q_2^*\mathcal{F} \to f_{W*}q_1^!\mathcal{G} \to p_{1*}f_Y^!\mathcal{G}.$$

Observe that if $S = \operatorname{Spec} A$, and we let the bottom row in 4.1.9 be $X_2 = W_2 = Y_2 = S$ with all the vertical maps being the structure morphism, then there the endomorphism of cohomological complexes $R\Gamma(X_1, \mathcal{F}) \to R\Gamma(Y_1, \mathcal{G})$ induced by T_{W_1} is exactly the same as the pushforward along the structure morphism $W_1 \to S$.

4.1.8. We can also define $*$ and !-pullbacks of cohomological correspondences. Again, assume that we have a commutative diagram

$$\begin{array}{ccccc} X_1 & \xleftarrow{q_2} & W_1 & \xrightarrow{q_1} & Y_1 \\ \downarrow{\scriptstyle f_X} & & \downarrow{\scriptstyle f_W} & & \downarrow{\scriptstyle f_Y} \\ X_2 & \xleftarrow{p_2} & W_2 & \xrightarrow{p_1} & Y_2 \end{array} \tag{4.1.9}$$

where the right square is derived Cartesian. Then, we can define the $*$-pullback of the bottom cohomological correspondence $T_{W_2} : p_2^* \mathcal{F} \to p_1^! \mathcal{G}$ as the composition:

$$f_W^* T_{W_2} : q_2^* f_X^* \mathcal{F} = f_W^* p_2^* \mathcal{F} \to f_W^* p_1^! \mathcal{G} \to q_1^! f_Y^* \mathcal{G},$$

where the last map comes from adjunction applied to the base change isomorphism along the right square. The induced maps on cohomology complexes are also compatible in the sense that we have a commutative square

$$\begin{array}{ccc} R\Gamma(X_2, \mathcal{F}) & \xrightarrow{T_{W_2}} & R\Gamma(Y_2, \mathcal{F}) \\ \downarrow & & \downarrow \\ R\Gamma(X_1, \mathcal{F}) & \xrightarrow{f_W^* T_{W_2}} & R\Gamma(Y_1, \mathcal{F}) \end{array}$$

Similarly, if the left square is derived Cartesian, then we can define the !-pullback of the bottom cohomological correspondence T_{W_2} as the composition:

$$f_W^! T_{W_2} : q_2^* f_X^! \mathcal{F} = f_W^! p_2^* \mathcal{F} \to f_W^! p_1^! \mathcal{G} = q_1^! f_Y^! \mathcal{G},$$

where here the first map follows from the base change isomorphism along the left square. We also hence have a commutative square

$$\begin{array}{ccc} R\Gamma(X_2, \mathcal{F}) & \xrightarrow{T_{W_2}} & R\Gamma(Y_2, \mathcal{F}) \\ \uparrow & & \uparrow \\ R\Gamma(X_1, \mathcal{F}) & \xrightarrow{f_W^* T_{W_2}} & R\Gamma(Y_1, \mathcal{F}) \end{array}$$

4.1.10. Let C, D be two algebraic correspondences given as follows,

$$\begin{array}{ccccccc} & & C & & & D & \\ & {\scriptstyle p_4}\swarrow & & \searrow{\scriptstyle p_3} \quad {\scriptstyle p_2}\swarrow & & \searrow{\scriptstyle p_1} & \\ X & & & Y & & & Z \end{array}$$

with cohomological correspondences $T_C : p_4^* \mathcal{F} \to p_3^! \mathcal{G}$ and $T_D : p_2^* \mathcal{G} \to p_1^! \mathcal{H}$. We can form the derived fiber product $C \times_Y^{\mathbb{L}} D$ which fits in the diagram

$$\begin{array}{ccccccc} & & & C \times_Y^{\mathbb{L}} D & & & \\ & & {\scriptstyle f}\swarrow & & \searrow{\scriptstyle g} & & \\ & C & & & & D & \\ {\scriptstyle p_4}\swarrow & & \searrow{\scriptstyle p_3} & & {\scriptstyle p_2}\swarrow & & \searrow{\scriptstyle p_1} \\ X & & & Y & & & Z \end{array}$$

If we let $q_2 = p_4 \circ f$, $q_1 = p_1 \circ q_2$, then we can define the composition as

$$f^* p_4^* \mathcal{F} \to f^* p_3^! \mathcal{G} \xrightarrow{(*)} g^! p_2^* \mathcal{G} \to g^! p_1^! \mathcal{H},$$

where $(*)$ is induced by adjunction from the proper base change isomorphism for coherent sheaves $g_* f^* \simeq p_2^* p_{3*}$. Note here that it is important that we take the derived fiber square for the base change isomorphism to hold in general.

4.2. **Hecke operators at $\mathfrak{p}$.** We now recall the construction of the Hecke operators acting on the Hilbert modular variety from [FP21, §4] and define, for each paritious weight $\kappa = (k_\sigma, k)$ the normalized Hecke operators $T_\mathfrak{p}^\kappa, S_\mathfrak{p}^\kappa$.

Consider the Hecke correspondence associated to the double coset

$$\mathrm{GL}_2(O_{F,\mathfrak{p}}) \begin{pmatrix} \mathfrak{p}^{-1} & \\ & 1 \end{pmatrix} \mathrm{GL}_2(O_{F,\mathfrak{p}}).$$

Observe that this corresponds to the Hilbert modular variety $\mathrm{Sh}_{\mathrm{Iw}_\mathfrak{p}}$ of parahoric level $\mathrm{Iw}_\mathfrak{p}$ as described in Section 3.4. We may define the naive $T_\mathfrak{p}$ Hecke operator as the composition

$$T_\mathfrak{p}^{naive} : p_2^* \omega^{\underline{\kappa}} \to p_1^* \omega^{\underline{\kappa}} \to p_1^! \omega^{\underline{\kappa}},$$

where the second map is given by tensoring with the fundamental class associated with p_1, which is constructed as in §4.1.2 since from Proposition 3.4.10, we see that the projection map p_1 is proper and lci.

However, we observe that the map $T_\mathfrak{p}^{naive}$ may not be optimally p-integral, and we may construct the normalized Hecke operator $T_\mathfrak{p}$ as follows. Let

$$N := \sum_{\tau \in \Sigma_\mathfrak{p}} \max\left(\frac{k_\tau + w}{2} - 1, \frac{w - k_\tau}{2} \right).$$

The map $p^N \cdot T_\mathfrak{p}^{naive}$ is clearly well defined over the generic fiber $\mathrm{Sh}^{\mathrm{tor}}_{\mathrm{Iw}_\mathfrak{p}, \mathbb{Q}_p}$, and in fact [FP21, Theorem 5.9] shows that this map extends integrally over $\mathrm{Sh}^{\mathrm{tor}}$. Thus, we have a map

$$T_\mathfrak{p} := p^N T_\mathfrak{p}^{naive} : p_2^* \omega^{\underline{\kappa}} \to p_1^! \omega^{\underline{\kappa}}.$$

Since we will need it later, let us give a more concrete description of $T_\mathfrak{p}$ over the open subscheme X_I° of the mod p fiber, using the local model.

We first observe the following from the description of the change-of-level maps between local models:

Proposition 4.2.1. *The map between local models $\pi_1 : U_{\{0,1\},\mu} \to M_{\{0\}}$ is locally an isomorphism, while the map $\pi_1 : U_{\{0,1\},\omega\mu} \to M_{\{0\}}$ is locally given by the relative Frobenius morphism. In particular, we see that in the first case the map on differentials $d\pi_1$ is the identity, while in the second case the map is zero.*

In particular, this allows us to calculate the normalization factors on each open X_I°, which corresponds through the local model diagram with $\prod_{\tau \in I} U_{0,1,\omega\mu} \prod_{\tau \notin I} U_{0,1,\mu}$.

Proposition 4.2.2. *The map $T_p|_{X_I^\circ}$ can be written in the form*

$$p^N \beta : p_2^* \omega^{\underline{\kappa}} \to p_1^! \omega^{\underline{\kappa}}$$

4.2.3. Moreover, following [FP21, §6.7], we also observe that the map of vector bundles $p_2^* \omega^{\underline{\kappa}} \to p_1^* \omega^{\underline{\kappa}}$ induced by the universal isogeny can also be described using a map of line bundles on the local model. More precisely, attached to $\underline{\kappa}$ we can define the line bundle over $M_{\{0\}}$, $\mathcal{L}_\kappa$. We have two maps from $M_{\{0,1\}}$ to $M_{\{0\}}$, namely the projection π_1 and the composition of the projection map to $M_{\{0\}}$ with the isomorphism to $M_{\{1\}}$, and we have the map

$$\alpha_\kappa^{naive} : \pi_2^*\mathcal{L}_\kappa \to \pi_2^*\mathcal{L}_\kappa,$$

which is locally given by multiplication by p, while on $U_{0,1,\omega\mu}$ it is given by the identity.

Let ξ denote the generic point of $U_{0,1,\mu}$. Over the completion $\hat{O}_{M_{\{0,1\}},\xi}$ at ξ, we see that the map $T_\mathfrak{p}$ can be written as a normalized isogeny map

$$\alpha_\kappa = p^{\kappa_\tau - 1}\alpha^{naive} : \pi_2^*\mathcal{L}_\kappa \to \pi_2^*\mathcal{L}_\kappa$$

since the projection map π_1 is an isomorphism. On the other hand, if we let ξ' denote the generic point of $U_{0,1,\mu}$, then we see that the map α_κ^{naive} is locally an isomorphism (indeed, it is in fact the Vershiebung map), and thus T_p can be written as the tensor product of α^{naive} with a map obtained from the inverse Cartier isomorphism. More precisely, we see that locally the map on differentials of the relative Frobenius map is given by $dx \mapsto pdx^{p-1}$, and we see that the Cartier isomorphism is given, by definition, locally as the map $dx \mapsto dx^{p-1}$. Moreover, by the Kodaira Spencer isomorphism, we have

$$\Omega^1_{\mathrm{Sh}} \simeq \oplus_\tau \omega_\tau^2,$$

and hence we see that we can write β as

$$\pi_2^*\mathcal{L}_\kappa \xrightarrow{\alpha_\kappa} \pi_1^*\mathcal{L}_\kappa = \pi_1^*\mathcal{L}_{\kappa-2} \otimes \pi_1^*\mathcal{L}_2 \xrightarrow{\mathrm{id}_{\pi_1^*\mathcal{L}_{\kappa-2}} \otimes \pi_1^* C^{-1}} \pi_1^*\mathcal{L}_{\kappa-2} \otimes \pi_1^!\mathcal{L}_2$$

4.3. **Composition of Hecke operators at $\mathfrak{p}$.** In order to gain a geometric understanding of composition of Hecke operators at $\mathfrak{p}$, we have to consider a larger space which parameterizes all isogenies, not just those of degree p^d.

Definition 4.3.1. Let $p - \mathrm{Isog}_\mathfrak{c}^{\mathrm{tor}}$ denote the moduli space over $\mathcal{M}_\mathfrak{c}^{\mathrm{tor}}$ parametrizing p-power isogenies of semi-abelian schemes respecting O_F-action.

By construction, we have projection maps $s, t : p - \mathrm{Isog}_\mathfrak{c}^{\mathrm{tor}} \to \mathcal{M}_\mathfrak{c}^{\mathrm{tor}}$.

Proposition 4.3.2. *The maps s, t are local complete intersection (lci) morphisms.*

Proof. For the open Shimura variety and the open scheme $p - \mathrm{Isog}$, this will be established in forthcoming work with Keerthi Madapusi [LM]. For the cusps, observe that since the p-power isogenies clearly preserve the cusp labels, we see that this follows from observing that all the cusps are ordinary, and hence locally around the cusps the maps s, t must also be lci. □

4.3.3. We now consider how composition of Hecke correspondences relates to composition of the support of the underlying algebraic correspondence. Recall from [LM] that we also construct the following derived composition map, which is a proper map

$$c : p - \mathrm{Isog} \times^{\mathbb{L}}_{\mathrm{Sh}} p - \mathrm{Isog} \to p - \mathrm{Isog}.$$

In particular, observe that we can think of the composition $T_\mathfrak{p} \circ T_\mathfrak{p}$ in the following way: Since $T_\mathfrak{p} \subset p - \mathrm{Isog}$ is a union of connected components, we may, via pushforward of the $T_\mathfrak{p}$ cohomological correspondence along the closed immersion $\iota : T_\mathfrak{p} \hookrightarrow p - \mathrm{Isog} \otimes \mathbb{F}_p$ (as in §?) obtain a cohomological correspondence supported on the diagram below, which we denote by

$T_{\mathfrak{p}}^{\sharp}$ to make it clear that we are looking at correspondences on the entire $p-\mathrm{Isog}\otimes\mathbb{F}_p$.

$$\begin{array}{ccc} & p-\mathrm{Isog}\otimes\mathbb{F}_p & \\ {\scriptstyle p_2}\swarrow & & \searrow{\scriptstyle p_1} \\ \mathcal{S} & & \mathcal{S} \end{array}$$

Note that by construction the cohomological correspondence $T_{\mathfrak{p}}^{\sharp}$ is zero outside the closed subscheme $T_{\mathfrak{p}}$. From §? The composition $T_p^{\sharp}\circ T_p^{\sharp}$ is hence associated with the diagram

$$\begin{array}{ccc} & p-\mathrm{Isog}\times^{\mathbb{L}}_{\mathrm{Sh}} p-\mathrm{Isog} & \\ {\scriptstyle p_2}\swarrow & & \searrow{\scriptstyle p_1} \\ \mathcal{S} & & \mathcal{S} \end{array}$$

If we now further pushforward $T_p^{\sharp}\circ T_p^{\sharp}$ along the composition map c, we see that the composition $c_*(T_p^{\sharp}\circ T_p^{\sharp})$, which we see must be supported only on the cycle-theoretic composition $T_{\mathfrak{p}}\cdot T_{\mathfrak{p}}$. In fact, this argument also shows the following:

Proposition 4.3.4. *The support of any $T_f\in\mathcal{H}_0(G)$ is the cycle $T_f\in\mathrm{Ch}^d(p-\mathrm{Isog})$.*

5. Higer Hida theory for Hilbert modular varieties

In this section, we will recall results of Boxer-Pilloni on a mod p control theorem for higher coherent cohomology of Hilbert modular varieties, and the associated set up. The results in this section are all due to Boxer-Pilloni [BP].

5.1. **$U_{p,I}$-operators.** Fix $I\subseteq\Sigma_{\mathfrak{p}}$. We first recall the construction of $U_{p,I}$-operators in regular cohomological weight. From the construction of $T_{\mathfrak{p}}$, we observe that for $I'\neq I$, over the open $X^{\circ}_{I'}$, the map is zero, because the normalization factor on $T_p|_{X_I^{\circ}}$ is strictly larger than N. Thus, we may assume that κ is such that it lies in the I-dominant Weyl chamber, namely for $\tau\notin I$, $\kappa_\tau\geq 2$, and for $\tau\in I$, $\kappa_\tau\leq 1$.

Observe that since T_p is zero outside X_I, we can apply the following result from [BP22, §2.1.3] to construct a cohomological correspondence supported on X_I.

Proposition 5.1.1. *Let X be a scheme and $\iota: Y\hookrightarrow X$ be a closed subscheme defined by a sheaf of ideals $\mathcal{I}\subseteq O_X$. For any quasi-coherent sheaf $\mathcal{F}$ over X, we let $\Gamma_Y(\mathcal{F})=\ker(\mathcal{F}\to\mathrm{Hom}(\mathcal{I},\mathcal{F}))$ be the subsheaf of sections with scheme theoretic support in Y. Let $f: X\to Z$ be a proper lci morphism of relative dimension zero such that the composition $g=\iota\circ f$ is also proper lci of relative dimension zero. Let $\mathcal{F}$ be a coherent sheaf on Z. Then $i_*g^!\mathcal{F}=\Gamma_Y f^!\mathcal{F}$.*

Note that in [BP22] this proposition is stated for finite flat morphisms, g,f, but the proof works equally well for lci morphisms of relative dimension zero.

5.2. **Geometric Jacquet-Langlands.** We now recall how the $U_{p,I}$ correspondence on the Hilbert modular variety, can be related to a U_p correspondence on a quaternionic Shimura variety that is of hyperspecial level at p. We will only discuss the case where $\#I\neq\#I^c$.

Firstly, we observe that the projection maps of the algebraic correspondence supported on X_I factor through closed subschemes. More precisely, we recall the following result from [ERX17a, Proposition 4.5].

Proposition 5.2.1. *Let $I_1^+ = \{\tau \in I : \sigma^{-1}\tau \notin I\}$. Then $\pi_1(X_I) = \mathcal{S}_{I_1^+}$.*

By replacing I with I^c, we see that the image under π_2 is $\mathcal{S}_{I_1^-}$, where $I_1^- = \{\tau \notin I : \sigma^{-1}\tau \in I\}$.

Observe that since the map p_1 factors through $\mathcal{S}_{I_1^+}$, and the map p_2 factors through $\mathcal{S}_{I_1^-}$, the cohomological correspondence supported on X_I induces a cohomological correspondence T_{X_I} from $(\mathcal{S}_{I_1^-}, p_2^*\omega^{\underline{k}})$ to $(\mathcal{S}_{I_1^+}, p_1^!\omega^{\underline{k}})$, i.e. we have a cohomological correspondence associated to the diagram

$$\begin{array}{ccc} & X_I & \\ {}^{q_2}\swarrow & & \searrow^{q_1} \\ \mathcal{S}_{I_1^-} & & \mathcal{S}_{I_1^+} \end{array}$$

We remark that the induced map on cohomology does *not* preserve the cohomological degree; this is not an issue.

5.2.2. We can apply the constructions in §4.1.8 to ! and $*$-pullback the cohomological correspondence, associated with the following commutative diagram:

$$\begin{array}{ccccc} \mathcal{S}_{I_1^-\cup I_1^+} & \xleftarrow{p_{2,1}} & X_{I,1} & \xrightarrow{p_{1,1}} & \mathcal{S}_{I_1^-\cup I_1^+} \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{S}_{I_1^-} & \xleftarrow{q_2'} & X_I' & \xrightarrow{q_1'} & \mathcal{S}_{I_1^-\cup I_1^+} \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{S}_{I_1^-} & \xleftarrow{q_2} & X_I & \xrightarrow{q_1} & \mathcal{S}_{I_1^+}. \end{array} \tag{5.2.3}$$

Here, we define X_I' to be the derived fiber product $q_1^{-1}(\mathcal{S}_{I_1^-\cup I_1^+})$, and similarly $q_2'^{-1}(\mathcal{S}_{I_1^-\cup I_1^+})$. By construction, the bottom right square is derived Cartesian, and so is the top left square. Now, we want to show that the schemes X_I' and $X_{I,1}$ are classical, and hence in fact simply the usual pullbacks. To see this, we note that it suffices to show that the underlying topological space of X_I' is a closed subspace of X_I of codimension $\#I_1^+$, while the underlying topological space of $X_{I,1}$ is a closed subspace of X_I of codimension $2\#I_1^+$. This is because both inclusions

$$X_{I,1}, X_I' \to X_I,$$

are quasi-smooth locally closed immersions of (possibly derived) schemes. Moreover, we note that the underlying classical schemes $X_{I,1}^{\mathrm{cl}}$ and $X_I'^{\mathrm{cl}}$ are simply the usual pullbacks, so it remains to calculate the dimension of this usual pullback. This is established in [BP], who show that the dimension of the usual fiber product $X_I \times_{\mathcal{S},p_1,p_2} X_I$ is d, hence $q_1^{-1}(\mathcal{S}_{I_1^-\cup I_1^+}) \cap q_2^{-1}(\mathcal{S}_{I_1^-\cup I_1^+})$ is of dimension $d - 2\#I_1^+$, as desired.

In fact, we observe that $X_{I,1}$ is a subspace of self-quasi isogenies between points on $\mathcal{S}_{I_1^-\cup I_1^+}$ whose kernel is of degree p^f.

5.2.4. If the projection maps $p_{1,1}, p_{2,1}$ are not generically finite, then we may consider the iterated correspondence $X_I^2 := X_I \times_{\mathcal{S},p_1,p_2} X_I$. As mentioned above, this will be of dimension d. More generally, we let

$$X_I^n := X_I \times_{\mathcal{S},p_1,p_2} \cdots \times_{\mathcal{S}} X_I,$$

this will also be of dimension d.

We can consider a similar pullback diagram as in (5.2.3) with $X_I = X_I^1$ replaced with X_I^n. We denote the projections by $p_{1,n}$ and $p_{2,n}$. Our first observation is the following theorem, which generalizes Proposition 5.2.1.

Theorem 5.2.5 ([BP]). *The closed subschemes $p_{1,n}(X_I^n)$ and $p_{2,n}(X_I^n)$ are generalized Goren-Oort strata (in the sense of Tian-Xiao* [TX19]*) of codimension $\#I$, and which intersect transversally. Moreover, there is some positive integer N, depending on I, such that*

$$p_{1,n}(X_I^n) = p_{1,N}(X_I^N) \qquad p_{2,n}(X_I^n) = p_{2,N}(X_I^N) \qquad \textit{for all } n \geq N.$$

Now consider $n \geq N$. Let $Y_I := p_{1,n}(X_I^n)$, and $Y_{I^c} := p_{2,n}(X_I^n)$. Let $Z_I := Y_I \cap Y_{I^c}$. Denote by $X_I'^n := p_{1,n}^{-1} Z_I$, and $Z_I^n := p_{2,n}'^{-1}(Z_I)$. We thus get a diagram

$$\tag{5.2.6} \begin{CD} Z_I @<{q_{2,n}}<< Z_I^n @>{q_{1,n}}>> Z_I \\ @VVV @VV{f}V @VVV \\ Y_{I^c} @<{p'_{2,n}}<< X_I'^n @>{p'_{1,n}}>> Z_I \\ @VVV @VV{g}V @VVV \\ Y_{I^c} @<{p_{2,n}}<< X_I^n @>{p_{1,n}}>> Y_I. \end{CD}$$

5.2.7. Moreover, we can also consider the commutative diagram of maps

$$\tag{5.2.8} \begin{CD} Z_I @>{\iota_{2,1}}>> Y_I \\ @V{\iota_{1,2}}VV @VV{\iota_1}V \\ Y_{I^c} @>{\iota_2}>> \mathcal{S} \end{CD}$$

Observe that this square is derived Cartesian. As in the case of $n = 1$, we can do base change along the rows of (5.2.6), since the bottom right and top left squares are derived Cartesian. If we $*$-pullback the bottom row to the second row, then !-pullback the second row to the top row, we get the following cohomological correspondence supported on the top row:

$$T_{Z_I^n} : q_{2,n}^* \iota_{1,2}^! \iota_2^* \omega^\kappa \to f^! p_{2,n}'^* \iota_2^* \omega = f^! g^* p_{2,n}^* \iota_2^* \omega \to f^! g^* p_{1,n}^! \iota_1^! \omega \to f^! p_{1,n}'^! \iota_{2,1}^* \iota_1^! \omega = q_{1,n}^! \iota_{2,1}^* \iota_1^! \omega.$$

Moreover, we observe that base change applied to (5.2.8) implies that we have an isomorphism $\iota_{2,1}^* \iota_1^! \omega \simeq \iota_{1,2}^! \iota_2^* \omega$. Moreover, if we let c be the codimension of Y_I in $\mathcal{S}$, with normal bundle $\mathcal{N}_{I,+}$, then we observe

$$\iota_{2,1}^* \iota_1^! \omega \simeq \iota_{1,2}^! \iota_2^* \omega \simeq \omega^\kappa|_{Z_I} \otimes \det \mathcal{N}_{I,+}[c]$$

is in fact a vector bundle in degree $-c$.

5.2.9. We now observe that we have a commutative diagram

$$\tag{5.2.10} \begin{array}{ccccc} R\Gamma(\mathcal{S}, \omega^\kappa) & \xrightarrow{\iota_2^*} & R\Gamma(Y_{I^c}, \iota_2^* \omega^\kappa) & \xleftarrow{\iota_{1,2!}} & R\Gamma(Z_I, \omega^\kappa|_{Z_I} \otimes \det \mathcal{N}_{I,+}[c]) \\ \Big\downarrow {\scriptstyle U_{p,I}^n} & & \Big\downarrow {\scriptstyle T_{X_I^n}} \quad \searrow & & \Big\downarrow {\scriptstyle T_{Z_I^n}} \\ R\Gamma(\mathcal{S}, \omega^\kappa) & \xleftarrow{\iota_{1!}} & R\Gamma(Y_I, \iota_1^! \omega^\kappa) & \xrightarrow{\iota_{2,1}^*} & R\Gamma(Z_I, \omega^\kappa|_{Z_I} \otimes \det \mathcal{N}_{I,+}[c]). \end{array}$$

Indeed, we see that by construction the left square is commutative. To show that the right square is commutative, we see that the composition $\iota_{2,1}^* \circ T_{X_I^n}$ is given by $g^* T_{X_I^n}$, while by construction $T_{Z_I^n}$ is the composition $g^* T_{X_I^n} \circ \iota_{1,2!}$.

5.2.11. We now want to show that the map $T_{Z_I^n}$ agrees with a $U^I_{p^n}$-operator acting on some quaternionic Shimura variety Z_I. To see this, we first observe that the underlying algebraic correspondence agrees:

Theorem 5.2.12 ([BP])**.** *Z_I is isomorphic over $\bar{\mathbb{F}}_p$ to the mod p fiber of a quaternionic Shimura variety $\mathrm{Sh}_{B_I,\bar{\mathbb{F}}_p}$ with the same level $K \subset \mathrm{GL}_2(\mathbb{A}_f)$, and the correspondence Z_I^n agrees with the closure in X_I^n of the correspondence given on the μ-ordinary locus Z_I° by the isogeny associated with the element given by*

$$b_I = \left(\begin{pmatrix} 1 & 0 \\ 0 & \mathfrak{p}^{-n} \end{pmatrix}, \dots, \begin{pmatrix} 1 & 0 \\ 0 & \mathfrak{p}^{-n} \end{pmatrix} \right) \in \prod_\tau \mathrm{GL}_2(\breve{F}_\mathfrak{p}).$$

Moreover, under the isomorphism $Z_I \simeq \mathrm{Sh}_{B_I,\bar{\mathbb{F}}_p}$, we have

(1) The vector bundle $\omega^\kappa|_{Z_I} \otimes \det_{\mathcal{N}_{I,+}}$ is isomorphic to a line subbundle of the automorphic vector bundle with weight $\kappa_I = (k'_\tau, w')$ where $w' = w - \#I$,

$$k'_\tau = \begin{cases} k_\tau & \text{if } \tau \notin J \\ k_\tau - 2 & \text{if } \tau \in J, \tau \notin I \\ -k_\tau & \text{if } \tau \in J, \tau \in I. \end{cases}$$

(2) The cohomological correspondence $T_{Z_I^n}$ agrees with $(U_p^I)^n$, where U_p^I denotes the classical U_p operator acting on coherent cohomology of Sh_{B_I} and regular weight k_I.

This quaternionic Shimura is associated with the quaternion algebra B over $\mathbb{Q}$ which is ramified only at a set of real places $J \subset \Sigma$ of even cardinality. J is defined in terms of a paranthesis diagram: We view $\Sigma_\mathfrak{p}$ as a cyclic set, and for each $\tau \in \Sigma_\mathfrak{p}$ is $\tau \in I$, we put a '$($' at the τ place, otherwise we put a '$)$'. Then the set J corresponds to the set of unmatched paranthesis in $\Sigma_\mathfrak{p}$.

Since we will use a similar argument in a later section, we sketch the main idea of the proof here. To check the equality of cohomological correspondences, it suffices to check over the μ-ordinary locus Z_I°. Moreover [BP] show that Z_I° in fact corresponds to a central leaf in $\mathcal{S}$, and the corresponding open $Z_I^{n,\circ} \subset Z_I^n$ in fact lies entirely in the open $X_I^{n,\circ}$ corresponding to the iterated product $X_I^\circ \times_\mathcal{S} \cdots \times_\mathcal{S} X_I^\circ$ of the open KR strata.

To see this, we first observe that we can understand $T_{Z_I^n}$ as the n-fold self composition of another cohomological correspondence $T_{Z_I^1}$, defined as follows. Starting with X_I, we may consider the correspondence obtained via sucessive iteration in the following sense: Set $r_{1,0} = p_1$, $r_{2,0} = p_2$, and define $\mathcal{S}_1 := r_{1,0}(X_I) \cap r_{2,0}(X_I)$, and $X_{I,1} := r_{1,0}^{-1}(\mathcal{S}_1) \cap = r_{2,0}^{-1}(\mathcal{S}_1)$. If the induced projection maps $r_{1,1}, r_{2,1} : X_{I,1} \to \mathcal{S}_1$ are not finite, then we may repeat this procedure and let $\mathcal{S}_2 := r_{1,1}(X_{I,1}) \cap r_{2,1}(X_{I,1})$, $X_{I,2} := r_{1,1}^{-1}(\mathcal{S}_2) \cap = r_{2,1}^{-1}(\mathcal{S}_2)$ and so on. Observe that for the same N as in Theorem 5.2.5, we have that the maps $r_{1,N}, r_{2,N}$ are finite, and moreover we see that $\mathcal{S}_N = Z_I$, $Z_I^N \simeq X_{I,N} \times_{Z_I} \cdots \times_{Z_I} X_{I,N}$, where we take the N-fold self product.

Thus, it remains to show that the cohomological correspondence $T_{X_{I,N}}$ induced on $X_{I,N}$ via successive $*$ and !-pullbacks and identifying Z_I with a quaternionic Shimura variey is the U_p^I-operator. To see this, we recall from §4.2.3 the local description of $U_{p,I}$ over the open X_I°. For simplicity we will explain this for subsets I such that $N = 1$ and the argument works in general. We further assume that $\#I < \#I^c$, since this entire construction commutes with taking Serre duals everywhere, and we have $Z_I = Z_{I^c}$. Moreover, under this assumption we have $I \subset J$. Let $Z_{I,N}^\circ$ denote the open preimage of Z_I° in $Z_{I,N}$. We first observe that we have a commutative diagram of local models

$$\begin{array}{ccccc} & & \tilde{P}_Z & & \\ & \swarrow & \downarrow & \searrow & \\ Z^\circ_{I,N} & & \tilde{P}_I & & \prod_{\tau\notin J} U_{0,1,\tau,\mu} \\ \downarrow & \swarrow & & \searrow & \downarrow \\ X^\circ_I & & & & \prod_{\tau\notin J} U_{0,1,\tau,\mu} \prod_{\tau\in J\backslash I} U_{0,1,\tau,\mu} \prod_{\tau\in I} U_{0,1,\tau,\omega\mu} \end{array}$$

where the left square is Cartesian. Let ξ be a generic point of X°_I, and ξ_I be a generic point of $Z^\circ_{I,N}$ mapping to ξ. The local model and the description of $T_{X^\circ_I}$ at ξ from §4.2.3 implies that the pullback $f^! g^* T_{X^\circ_I}$ restricted to ξ_I is simply given by

$$\prod_{\tau\notin J} \alpha_\tau \otimes \det \mathcal{N}_{I,+}|_{\xi_I}.$$

Note that $\mathcal{N}_{I,+}|_{\xi_I}$ is a one-dimensional subbundle of the trivial rank 2 automorphic vector bundle $\mathcal{L}_\tau|_{\xi_I}$, for $\tau \in J\backslash I$, and hence $U^I_p|_{\xi_I}$ is an isomorphism on the τ-part.

We remark that the normalization factors for $U_{p,I}$ and U^I_p are still the same as the extra factor of $p^{\#I}$ is subsumed into the action on the determinant bundles ϵ_τ.

5.2.13. As a consequence of the commutative diagram (5.2.10), and compatibility of $U^n_{p,I}$-operators, we see that we have a commutative diagram

$$\begin{array}{ccc} R\Gamma(X, \omega^{\underline{\kappa}}) & \longrightarrow & R\Gamma(Z_I, \omega^{\underline{\kappa}_I}) \\ \downarrow{\scriptstyle U_{p,I}} & & \downarrow{\scriptstyle U^I_p} \\ R\Gamma(X, \omega^{\underline{\kappa}}) & \longrightarrow & R\Gamma(Z_I, \omega^{\underline{\kappa}_I}) \end{array}$$

Moreover, these maps induce an isomorphism of ordinary parts, and we have the following mod p control theorem:

Theorem 5.2.14 ([BP])**.** *Suppose that the weight κ satisfies for all $\tau \notin J$ $k_\tau \leq -1$, or $k_\tau \geq 3$. Then if $\#I < \#I^c$, we have isomorphisms*

$$e(U_{p,I})R\Gamma(X, \omega^{\underline{\kappa}}) \simeq e(U^I_p)R\Gamma(Z_I, \omega^{\underline{\kappa}_I}) \simeq e(U^I_p)R\Gamma(Z^\circ_I, \omega^{\underline{\kappa}_I}),$$

while if $\#I > \#I^c$, we have isomorphisms

$$e(U_{p,I})R\Gamma(X, \omega^{\underline{\kappa}}) \simeq e(U^I_p)R\Gamma(Z_I, \omega^{\underline{\kappa}_I}) \simeq e(U^I_p)R\Gamma_c(Z^\circ_I, \omega^{\underline{\kappa}_I}),$$

where $R\Gamma_c(Z^\circ_I, —)$ is cohomology of Z_I with compact support as defined in Hartshorne [Har71].

This result implies that in the case $\#I < \#I^c$, we can restrict a class to the open Z°_I to show that it is ordinary.

6. Weight-shifting

In this section, we will show that given a cohomological class v in $H^i(\mathcal{S}, \omega^1)$, there exists some set of partial θ and Hasse operators such that we can shift v into a cohomology group with regular weight. We will then use this set of weight shifting operators to determine which $U_{p,I}$ operator to apply to v.

To see this we first recall the Cousin complex, introduced in [BCGP21, §4.2.30].

6.1. **Cousin Complex.**

6.1.1. Let S be a smooth scheme over a field k and let $\mathcal{L}$ be an invertible sheaf on S. We assume that we have n non-vanishing sections of line bundles $s_i \in H^0(S, \mathcal{L}_i)$, and we set $\mathcal{L} = \otimes_{i=1}^n \mathcal{L}_i$.

We denote the divisors $D_i = V(s_i)$, and note that D_i are effective Cartier divisors on S. Set $s = \prod_{i=1}^d s_i$. Set $D = V(s) = \cup_i D_i$. We further assume that $D = \cup_i D_i$ is a strict normal crossing divisor on S.

For all m, consider the following exact complex of coherent sheaves on S:

$$0 \to O_S \xrightarrow{s^m} \mathcal{L}^m \to \bigoplus_{i=1}^n \mathcal{L}^m/(s_i^m) \to \bigoplus_{1\le i<j\le n} \mathcal{L}^m/(s_i^m, s_j^m) \to \dots \mathcal{L}^m/(s_1^m, \dots, s_d^m) \to 0,$$

where for $0 \le k \le n$, the object placed in degree $k+1$ is

$$\bigoplus_{1\le i_1<\dots<i_k\le n} \mathcal{L}^m/(s_{i_1}^m, \dots, s_{i_k}^m).$$

Here, when we write $\mathcal{L}^m/(s_{i_1}^m, \dots, s_{i_k}^m)$, we mean the quotient $\mathcal{L}^m/\mathcal{L}^m(s_{i_1}^m\mathcal{L}_{i_1}^{-m}, \dots, s_{i_k}^m L_{i_k}^{-m})$. The differential maps take a section $(f_{i_1,\dots,i_k})_{1\le i_1<\dots<i_k\le n}$ to the section

$$\left(\sum (-1)^{i_j} \overline{f_{i_1,\dots,\hat{i}_j,\dots,i_{k+1}}}\right)_{1\le i_1<\dots<i_{k+1}\le n}$$

where $\overline{f_{i_1,\dots,\hat{i}_j,\dots,i_{k+1}}}_{1\le i_1<\dots<i_{k+1}\le n}$ is the class modulo s_j^m of $f_{i_1,\dots\hat{i}_j,\dots,i_{k+1}}$.

We then have a commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & O_S & \longrightarrow & \mathcal{L}^{m+1} & \longrightarrow & \oplus_{i=1}^n \mathcal{L}^{m+1}/(s_i^{m+1}) & \longrightarrow & \dots \\ & & \uparrow{\scriptstyle \mathrm{id}} & & \uparrow{\scriptstyle s} & & \uparrow{\scriptstyle s} & & \\ 0 & \longrightarrow & O_S & \longrightarrow & \mathcal{L}^m & \longrightarrow & \oplus_{i=1}^n \mathcal{L}^m/(s_i^m) & \longrightarrow & \dots \end{array}$$

Passing to the limit over m, we get the following exact complex:

$$\begin{aligned} 0 \to O_S \to \varinjlim_m \mathcal{L}^m \to \varinjlim_m \bigoplus_{i=1}^n \mathcal{L}^m/(s_i^m) \to \varinjlim_m \bigoplus_{1\le i<j\le n} \mathcal{L}^m/(s_i^m, s_j^m) \to \dots \\ \to \varinjlim_m \mathcal{L}^n/(s_1^m, \dots, s_d^m) \to 0 \end{aligned}$$

where in all the direct limits, the transition maps are given by multiplication by powers of s.

6.1.2. We now specialize to the situation where $S = \mathcal{S}$, the mod p fiber of the Hilbert modular variety, and the s_i are the d partial Hasse invariants. Observe that D_i is hence the Goren-Oort strata $\mathcal{S}_\tau$. The sheaf $\mathcal{L}$ is thus the Hodge bundle $\otimes_\tau \omega_\tau$. For a set $T \subset \Sigma$, we let

$$\mathcal{S}_T^\circ = \mathcal{S}_T \setminus \cup_{T'\supset T} \mathcal{S}_{T'}$$

which is also the associated Ekedahl-Oort stratum. Ekedahl-Oort strata in the minimal compactification are known to be affine, following [Box15] and [GK19].

We thus observe that for any subcanonical automorphic vector bundle $\mathcal{V}(D)$, the sheaves $\mathcal{V}(D) \otimes \mathcal{L}|_{\mathcal{S}_T^\circ}$ are acyclic, and thus have vanishing higher sheaf cohomology. For every sheaf $\mathcal{V}$, the Cousin complex is thus defined to be the following complex, which we see will be quasi-isomorphic to $R\Gamma(\mathcal{S}, \mathcal{V}(D))$.

$$\varinjlim_m H^0(\mathcal{L}^m \otimes \mathcal{V}(D)) \to \varinjlim_m \bigoplus_{i=1}^n H^0(\mathcal{L}^m \otimes \mathcal{V}(D)/(s_i^m)) \to \varinjlim_m \bigoplus_{1 \leq i < j \leq n} H^0(\mathcal{L}^m \otimes \mathcal{V}(D)/(s_i^m, s_j^m)) \to \dots$$
$$\to \varinjlim_m H^0(\mathcal{L}^m \otimes \mathcal{V}(D)/(s_1^m, \dots, s_d^m))$$

We now specialize the the case where $\mathcal{V}$ is the vector bundle of weight one. We will consider $v \in H^j(\mathcal{S}, \omega^1(D))$, which we assume to be Eisenstein, and thus we can freely apply Serre duality.

Proposition 6.1.3. *Let $v \in H^j(\mathcal{S}, \omega^1(D))$. There is some subset T of size $d-j$ (depending on v), such that if we take $h_T := \prod_{\tau \in J} h_\tau$, the image $h_T(v)$ is non-zero.*

Proof. Our first observation is that, from the Cousin complex, there is some minimal positive integer m such that we may represent v as an element

$$(f_{T'}) \in \bigoplus_{T' = \{\tau_1, \dots, \tau_j\} \subset \Sigma} H^0(\mathcal{L}^m/(h_{\tau_1}^m, \dots, h_{\tau_j}^m)).$$

Let T' be some subset such that $f_{T'} \neq 0$. We claim that we can take $T = \Sigma \backslash T'$. To see this, let $Y = \mathcal{S} \backslash (\cup_{\tau \in T'} \mathcal{S}_\tau)$. We may consider the Cousin complex associated with this scheme Y and we also let the sections s_i be the partial Hasse invariants in T'. We want to show that the image of v under the restriction

$$H^j(\mathcal{S}, \omega^1) \to H^j(Y, \omega^1|_Y)$$

is non-zero, this amounts to showing that there is no element in

$$\bigoplus_{\tau_i} H^0(\mathcal{L}^m/(h_{\tau_1}^m, \dots, \hat{h}_{\tau_i}^m, \dots, h_{\tau_j}^m))$$

whose image under the differential maps is $f_{T'}$. Observe that if such an element existed, then we would be able to express v as some

$$(f'_{T'}) \in \bigoplus_{T' = \{\tau_1, \dots, \tau_j\} \subset \Sigma} H^0(\mathcal{L}^m/(h_{\tau_1}^m, \dots, h_{\tau_j}^m)),$$

where $f'_{T'} = 0$. However, such a property cannot hold for all subsets T' of size j, unless $v = 0$. □

In fact, we can also show the following:

Proposition 6.1.4. *For the same set T as in Proposition 6.1.3, we have that v lies in the image of the map*

$$H^j(\mathcal{S}, \mathcal{L}_T^{-1} \otimes \omega^1) \to H^j(\mathcal{S}, \omega^1).$$

Proof. This follows from duality, assuming the class v is Eisenstein. □

6.2. **Partial Theta operators.** We recall the construction of various theta operators acting on coherent sheaves mod p following [EFG+21] (modified to the current setting of Hilbert modular varieties), whose restrictions to various Goren-Oort strata are, up to a constant depending on the automorphic weight, the theta operators defined in [ERX17a].

6.2.1. Over the locus given by the non-vanishing of h_τ, we can construct a splitting of the Hodge filtration

$$0 \to \omega_\tau \to \mathcal{H}_\tau \to \mathrm{Lie}(A^\vee)_\tau \to 0$$

by considering the composition of the maps

$$H_\tau \xrightarrow{V_\tau} \omega^p_{\sigma^{-1}\tau} \xrightarrow{h_\tau^{-1}} \omega_\tau,$$

where the non-vanishing of h_τ allows us to construct the last inverse map.

Recall that we have the Gauss-Manin connection (which decomposes according to the O_F-action),

$$\nabla_\tau : \mathrm{Sym}^k(\mathcal{H}_\tau) \to \mathrm{Sym}^k(\mathcal{H}_\tau) \otimes \Omega^1_\tau$$

and by Griffiths transversality we see that

$$\nabla(\omega_\tau) \subseteq F^{k-1}(\mathrm{Sym}^k(\mathcal{H}_\tau)) \otimes \Omega^1_\tau,$$

where $F^{k-1}(\mathrm{Sym}^k(\mathcal{H}_\tau)) = \mathrm{im}(\omega^{k-1} \otimes \mathcal{H}_\tau \to \mathrm{Sym}^k(\mathcal{H}_\tau)$ is the $k-1$-th piece of the Hodge filtration. Moreover, we can apply the Kodaira-Spencer isomorphism to get

$$\Omega^1_\tau \simeq \omega^2_\tau.$$

We define θ_τ for $k \geq 1$ as

$$\theta_\tau : \omega^k_\tau \hookrightarrow \mathcal{H}_\tau \xrightarrow{\nabla_\tau} F^{k-1}(\mathrm{Sym}^k(\mathcal{H}_\tau)) \otimes \Omega^1_\tau \xrightarrow{h_\tau} \omega^{k-1}_\tau \otimes \omega^p_{\sigma^{-1}\tau} \otimes \omega^2_\tau,$$

Proposition 6.2.2. *The restriction $\theta_\tau|_{\mathcal{S}_\tau}$ is k_τ times the θ_τ operator defined using the Kodaira-Spencer isomorphism in* [ERX17b]. *Thus, if $p \nmid k_\tau$, we have θ_τ is an isomorphism of vector bundles on $\mathcal{S}_\tau$. Otherwise θ_τ is zero.*

Proof. This follows by observing that when we restrict ∇_τ to $\mathcal{S}_\tau$, this is after tensoring with h_τ exactly the Kodaira-Spencer isomorphism on S_τ. Thus, θ_τ being an isomorphism follows exactly as in [ERX17b]. □

Proposition 6.2.3. *θ_τ commutes with multiplication by the Hasse invariant $h_{\tau'}$ for all τ, τ'.*

The key result we need is the following:

Proposition 6.2.4. *Let v be as in Proposition 6.1.3. Then for any $\tau \in T$, we have $\theta_\tau(v) \neq 0$,*

Proof. We want to show that if $h_\tau(v) \neq 0$, then $\theta_\tau(v) \neq 0$. As in the proof of Proposition 6.1.3, we may assume that T' is such that $f_{T'} \neq 0$. We want to show that for $\tau \in T$, we have $\theta_\tau(f'_T) \neq 0$, to see this, it in fact suffices from Proposition 6.2.2 to show that the restriction of $f_{T'}$ to $\mathcal{S}_{T' \cup \tau}$ is nonzero.

In order to show this, we will show that $f_{T'}$ cannot be divided by h_τ. We will in fact make a stronger claim: that the cohomology group $H^0(\mathcal{S}_{T'}, \mathcal{L}_\tau^{-1} \otimes \mathcal{L}_{T'} \otimes \omega^1)$ vanishes. This follows from the main theorem of [Kos23]. Indeed, we see that for each $\tau \in T$, we see that the ample cone has a a hyperplane equation of the form

$$\cdots + p^f k_{\sigma^{-1}\tau} \pm k_\tau + \ldots,$$

where the coefficients of all other $k_{\tau'}$ have absolute value in the set $\{p, \ldots, p^{f-1}\}$, and satisfies that for $\tau \in T'$, the sign of the coefficient of k_τ and $k_{\sigma^{-1}\tau}$ are different. Thus, we see that for the automorphic weight corresponding to $\mathcal{L}_\tau^{-1} \otimes \mathcal{L}_{T'} \otimes \omega^1$, it does not lie in the ample cone, and thus has no sections. □

In particular, we have the following corollary:

Corollary 6.2.5. *For any $v \in H^i(\mathcal{S}, \omega^1)$, there is a set I such that for all $\tau \in I$, we have $\theta_\tau(v), h_\tau(v) \neq 0$, and for all $\tau \in I^c$, we have that v lies in the image of θ_τ and h_τ.*

6.2.6. For any subset I, we can thus consider the sequence of partial theta and partial Hasse operators defined as follows. We may partition the set Σ as follows. If

$$\begin{aligned}
\tau \notin \Sigma_{\mathfrak{p}} &\qquad \text{set } \tau \in I_{h,+} \\
\tau \in I, \sigma^{-1} \notin I &\qquad \text{set } \tau \in I_{\theta,+} \\
\tau \in I, \sigma^{-1} \in I &\qquad \text{set } \tau \in I_{h,-} \\
\tau \in \Sigma_{\mathfrak{p}} \backslash I, \sigma^{-1} \notin I &\qquad \text{set } \tau \in I_{h,+} \\
\tau \in \Sigma_{\mathfrak{p}} \backslash I, \sigma^{-1} \in I &\qquad \text{set } \tau \in I_{\theta,-}.
\end{aligned}$$

Thus, for every $v \in H^j(\mathcal{S}, \omega^1)$, there exists some set I such that v is non-zero under the maps

$$H^j(\mathcal{S}, \omega^1) \to H^j(\mathcal{S}, \omega^{1+I_{\theta,+}+I_{h,+}}),$$

and lies in the image of

$$H^j(\mathcal{S}, \omega^{1-I_{\theta,-}-I_{h,-}}) \to H^j(\mathcal{S}, \omega^1),$$

where we abuse notation to denote the weight $I_{\theta,+}$ as the sum of weights $(\dots, p, 1, \dots)$ with 1 in the τ place and p in the $\sigma^{-1}\tau$ place for $\tau \in I_{\theta,+}$ and similarly for $I_{h,+}$.

We now want to study the following composition

$$T_{\mathfrak{p},I} : R\Gamma(\mathcal{S}, \omega^{1-I_{\theta,-}-I_{h,-}}) \to R\Gamma(\mathcal{S}, \omega^1) \xrightarrow{T_{\mathfrak{p}}} R\Gamma(\mathcal{S}, \omega^1) \to R\Gamma(\mathcal{S}, \omega^{1+I_{\theta,+}+I_{h,+}}),$$

which we want to show is supported on the KR-strata X_I.

Proposition 6.2.7. *The composition $T_{\mathfrak{p},I}$ has support only on X_I.*

Proof. Consider any subset $J \subset \Sigma_{\mathfrak{p}}$. Our first observation is that if we let $p_1(X_J) = \mathcal{S}_{J^+}$, then we must have $J^+ \cap (I_{\theta,+} \cup I_{h,+}) = \emptyset$. Observe that we know that the map restricted to X_J factors through the pushforward map

$$R\Gamma(\mathcal{S}_{J^+}, p_1^! \omega^1) \to R\Gamma(\mathcal{S}, \omega^1),$$

and observe that the automorphic weight of $p_1^!\omega^1$ satisfies $k_\tau = 0$ for $\tau \in J^+$, hence we see will from Proposition 6.2.2 that if $\tau \in \mathcal{S}_{J^+} \cap I_{\theta,+}$, then the composition

$$R\Gamma(\mathcal{S}_{J^+}, p_1^! \omega^1) \to R\Gamma(\mathcal{S}, \omega^1) \xrightarrow{\theta_\tau} R\Gamma(\mathcal{S}, \omega^{1+\theta_\tau})$$

is simply zero. We similarly see that we cannot have $\tau \in \mathcal{S}_{J^+} \cap I_{\theta,+}$, since the sequence

$$R\Gamma(\mathcal{S}_{J^+}, p_1^! \omega^1) \to R\Gamma(\mathcal{S}, \omega^1) \xrightarrow{h_\tau} R\Gamma(\mathcal{S}, \omega^{1+h_\tau})$$

is exact.

Moreover, we can also look at the generic point ξ of each X_J. Observe that the pullback of the θ_τ operator to ξ is given by the pullback of the differential operator ∇_τ. Moreover, observe that if we have both $\tau \in J$ and $\sigma^{-1}\tau \in J$ then locally on X_J the isogeny is exactly given by the partial Frobenius at τ, hence the composition is zero on the generic point.

In particular, we can look at all four possibilities for $\tau, \sigma^{-1}\tau$ being in J or not in J, and we see that for T_I to have support on J, we must have $\sigma^{-1}\tau$ not in J for all $\tau \in I_{\theta,+}$. Otherwise, we see that if $\sigma^{-1}\tau \in J$ and $\tau \notin J$, then $\tau \in J^+$. Moreover, we see that we must also have $\sigma^{-1}\tau$ not in J for all $\tau \in I_{h,+}$, because we see that if $\tau \in I_{h,+}$, then $\sigma(\tau) \in I_{h,+} \cup I_{\theta,+}$, and we can work successively from the $\sigma^r(\tau)$ which is in $I_{\theta,+}$.

Thus, we see that J satisfies for all $\tau \in I_{h,+} \cup I_{\theta,+}$, $\sigma^{-1}\tau \notin J$. Taking duals, we can make the same argument that for all $\tau \in I_{h,-} \cup I_{\theta,-}$, $\sigma^{-1}\tau \in J$. This exactly determines J, and we must have $J = I$. □

We want to prove a similar statement for T_p^2 or other iterated compositions, to do this, we need to be able to describe the components in more detail, which we will do so in the next section

7. Composition of Hecke correspondences

In this section, we recall results from [Lee21] about the composition of algebraic cycles underlying Hecke operators.

7.1. **Irreducible components of $p - \mathrm{Isog}$.** We first recall how to describe irreducible components of the special fiber of $p - \mathrm{Isog}$, following [Lee21, §6.2]. We say that an irreducible component C of $p - \mathrm{Isog} \otimes \mathbb{F}_p$ is $[b]$-dense if the Newton strata $C^{[b]}$ is dense in C.

We have constructed in loc. cit. a map

$$\tilde{\pi}_\infty : \mathrm{RZ}(G, b, \mu)^{red} \times J_\infty^{(p^{-\infty})} \times \mathrm{RZ}(G, b, \mu)^{red} \to p - \mathrm{Isog} \otimes \kappa^{[b]}$$

where $J_\infty^{(p^{-\infty})}$ is a perfection of the Igusa variety as defined by Mantovan.

Proposition 7.1.1 ([Lee21, Prop 6.2.3])**.** *Every irreducible $[b]$-dense component of $p - \mathrm{Isog} \otimes \mathbb{F}_p$ is the image of some triple (X, Y, Z), where $X, Z \subset \mathrm{RZ}(G, b, \mu)^{red}$ are irreducible components, and $Y \subset J_\infty^{(p^{-\infty})}$ is an irreducible component. Moreover, the pair (X, Z) is determined up to the action of $J_b(\mathbb{Q}_p)$.*

In particular, given a $[b]$-dense irreducible component C and an element $j \in J_b(\mathbb{Q}_p)$, we can associate another $[b]$-dense irreducible component which we denote C_b, defined as follows. Let (X, Y, Z) be a triple whose image under $\tilde{\pi}_\infty$ is C, then we define C_j to be the image of $(X, Y, j \cdot Z)$, where we recall that $J_b(\mathbb{Q}_p)$ acts on $\mathrm{RZ}(G, b, \mu)^{red}$ by premultiplication by j. We further recall the following theorem, which allows us to understand the structure of compositions of components with an element of the Hecke algebra:

Theorem 7.1.2 ([Lee21, Theorem 7.4.9])**.** *Let $f \in \mathcal{H}(G(\mathbb{Q}_p)//K_p, \mathbb{Q})$. Then the restriction to $[b]$-dense irreducible components of the composition $C \cdot h(f)$ consists of terms of the form $\tilde{\pi}_\infty(X, Y, j \cdot Z)$, where $j \in J_b(\mathbb{Q}_p)$ is determined by the image of f under the twisted Satake morphism.*

As a corollary, we see that when specialized to the Hilbert modular variety, we have:

Proposition 7.1.3. *Let C be a $[b]$-dense irreducible component in $p - \mathrm{Isog} \otimes \mathbb{F}_p$. Then any $[b]$ dense irreducible component in $C \cdot T_{\mathfrak{p}}$ is of the form $C_{\mu(p)}$ or $C_{\omega\mu(p)}$.*

In fact, for Hilbert modular varieties the proof of Theorem 7.1.2 can in fact be adapted to show a more refined result, about $[b']$-dense components for $[b'] \leq [b]$. Note if C is $[b]$ dense, then the Newton strata appearing in any composition $C \cdot D$ must be $[b']$-dense for $[b'] \leq [b]$.

Proposition 7.1.4. *Let C be a $[b]$-dense irreducible component in $p-\mathrm{Isog}\otimes\mathbb{F}_p$. Then any $[b']$-dense irreducible component in $C\cdot T_{\mathfrak{p}}$ is of the form $C'_{(\mu)(p)}$ or $C'_{\omega(\mu)}$, where C' is an irreducible component with $\pi_1(C')\subset\pi_1(C)$, and $\pi_2(C')\subset\pi_2(C)$.*

Proof. It is clear that $\pi_2(C')\subset\pi_2(C)$, we will show this for π_1.

We first observe that the Newton strata in $\mathcal{S}$ can be totally ordered, and thus we may enumerate the unramified strata $[b']\leq[b]$ as $[b]=[b_0]<[b_1]<\cdots<[b_n]$. We will induct on n. Consider some irreducible component A appearing in $C\cdot T_{\mathfrak{p}}$ which is $[b_1]$-dense, and suppose that it is the image under $\tilde{\pi}$ of the form (X_1,Y_1,Z_1). We then look at the image of $[X_1]$ under the cohomological correspondence u_C - this has to be the union of some irreducible components of $\mathrm{RZ}(G,b,\mu)^{\mathrm{red}}$ whose images are all contained in $\pi_1(C)$. We know that this is true also for the composition $u_{C\cdot T_p}$. Thus, we see that if $C\cdot T_{\mathfrak{p}}$ contained a $[b_1]$ dense component C' whose image $\pi_1(C')$ was not contained in $\pi_1(C)$, then this would imply that the action of the composition $u_{C\cdot T_p}$ on $[Z_1]$ would be some formal sum of components, one of which is not contained in $\pi_1(C)$, since there cannot be cancellation among the terms which are $[b_1]$-dense, hence we get a contradiction. □

In fact, for the Hilbert modular variety we can say that such components must always appear as a transverse intersection.

Proposition 7.1.5. *Suppose that C is a component of T_p corresponding to I. Then the components appearing in $C\cdot T_p$ are all of the form $C\cdot C'$, where C' is an irreducible component of T_p such that $p_1(C)$ intersects transversely with $p_2(C)$.*

Proof. By Proposition 7.1.4, we see since $p_1(X_I)=\mathcal{S}_{I_1^+}$, that we only need to consider products $C\cdot C'$ with $C'=X_{I'}$ for a set I' with $I_1'^+\supset I_1^+$. Thus, we see that $I_1^{c,+}$ and I_1^+ are disjoint, which implies that $p_1(C)$ intersects transversely with $p_2(C)$. □

7.2. **The operator T_{p^n}.** We now want to understand the irreducible components appearing in the underlying algebraic cycle of the Hecke operator T_{p^n}, which we define for $n=2$ as

$$T_{p^2}=T_p\cdot T_p-2p^d\cdot 1_{pK_p}.$$

is inductively defined for $n>2$ to be

$$T_{p^n}=T_{p^{n-1}}\cdot T_p-p^d(T_{p^{n-2}}\cdot 1_{pK_p}).$$

Proposition 7.2.1. *The cycle underlying T_{p^n} is effective.*

Proof. We will show that T_{p^n} corresponds to an effective cycle over the generic fiber, from which the claim over the special fiber follows. To see this, we see that for $n=2$, we can write T_p as a sum of left K_p-cosets $T_p=1_{\mu(p)K_p}+1_{\omega(\mu)(p)K_p}+\sum 1_{n_i\omega(\mu)(p)K_p}$ for some $n_i\in N(\mathbb{Q}_p)$, from which we see that the function T_{p^2} can be written as a sum of left cosets with positive coefficients, and more precisely can be written as

$$T_{p^2}=1_{\mu^2(p)K_p}+\sum 1_{n_i\omega(\mu)^2(p)K_p},$$

where we are summing over p^{2d}-terms of the form $1_{n_i\omega(\mu)^2(p)K_p}$ for some $n_i\in N(\mathbb{Q}_p)$.

Now, we can proceed by induction on n. To see this, we will show that we have

$$T_{p^n}=1_{\mu^n(p)K_p}+\sum 1_{n_i\omega(\mu)^n(p)K_p},$$

where we are summing over p^{nd}-terms of the form $1_{n_i\omega(\mu)^2(p)K_p}$ for some $n_i \in N(\mathbb{Q}_p)$. This is true for $n = 1, 2$, and we can expand the expression for $T_{p^{n-1}} \cdot T_p$ to verify this for T_{p^n}. Thus, we see that T_{p^n} is the sum of K_p-double cosets with positive coefficients, from which the claim follows.

□

In particular, this proposition shows that there is no cancellation of cohomological correspondences occuring.

We now need to describe the cycles appearing in T_{p^n}, and we want to show that applying the partial theta operators and Hasse invariants as in Proposition 6.2.7 puts a strong restriction on the components which can appear in its support. Define the composition

$$T_{\mathfrak{p}^n,I} : R\Gamma(\mathcal{S}, \omega^{1-I_{\theta,-}-I_{h,-}}) \to R\Gamma(\mathcal{S}, \omega^1) \xrightarrow{T_{\mathfrak{p}^n}} R\Gamma(\mathcal{S}, \omega^1) \to R\Gamma(\mathcal{S}, \omega^{1+I_{\theta,+}+I_{h,+}}),$$

Theorem 7.2.2. *Let $n \geq N$. The irreducible components C appearing in the support of $T_{\mathfrak{p}^n,I}$ all satisfy $\pi_1(C) = Y_I$, $\pi_2(C) = Y_{I^c}$.*

Proof. We will proceed by induction, since the case of $n = 1$ is simply Proposition 6.2.7. Recall from the proof that result that there are four for a component C in which composition with partial theta maps or Hasse invariants will result in a zero map:

(1) $p_1(C) \subseteq \mathcal{S}_{I_{\theta,+} \cup I_{h,+}}$
(2) $p_2(C) \subseteq \mathcal{S}_{I_{\theta,-} \cup I_{h,-}}$
(3) For all $\tau \in I_{\theta,+}$, the isogeny over C contains the partial Frobenius at τ.
(4) For all $\tau \in I_{\theta,-}$, the dual of the isogeny over C contains the partial Frobenius at τ.

To see this, we now consider the case of a component appearing in T_{p^n}. Our first observation is that from Proposition 7.1.5, we see that C can be written as the composition

$$C_1 \cdot C_2 \dots C_n,$$

where each C_i is some component X_{I_i}, and we have inclusions

$$I_{1,1}^+ \subseteq I_{2,1}^+ \subseteq \dots I_{n,1}^+.$$

Let $C' = C_1 \dots C_{n-1}$. Observe that if C_1 satisfies either (1) or (2), then we must have, by Proposition 7.1.4 that C will also satisfy (1) or (2). Hence, we see that for $\tau \in I_{\theta,+} \cup I_{h,+}$, we cannot have both $\tau \notin I_1$ and $\sigma^{-1}\tau \in I_1$, and similarly for $\tau \in I_{\theta,-} \cup I_{h,-}$, we cannot have both $\tau \in I_1$ and $\sigma^{-1}\tau \notin I_1$.

To simplify some of the combinatorics, we now represent subsets I using a parenthesis diagram, as introduced in [BP], and defined as follows. For each $\tau \in I$, we put a '(' at the τ position, and we match '(' with the first ')' appearing in $\sigma(\tau), \sigma^2(\tau), \dots$ and so on. We label each τ with the minimal number of matched parenthesis between τ and its pair.

Suppose that C' satisfies (3). Observe that as long as C_n satisfies that the isogeny at τ is not the partial Vershiebung at τ (i.e. if both $\tau, \sigma^{-1}\tau \notin I$), then the composition will also satisfy (3). A similarly argument holds if C' satisfies (4), hence as long as C_n does not satisfy that the isogeny at τ is the partial Frobenius at τ (i.e. if both $\tau, \sigma^{-1}\tau \in I$), then the composition will also satisfy (4).

We now suppose that C' satisfies (3) or (4). Observe that we have two possibilities for an irreducible component C lying in the product $C' \cdot C_n$: either

(a) C is $[b]$-dense, where either C' or C_n is $[b]$-dense

(b) C is $[b']$-dense, where neither C' or C_n is $[b']$-dense

In case (a), we claim that we can express the product $C' \cdot C_n$ in the form C''_j for some $j \in J_b(\mathbb{Q}_p)$ and some component C''. We want to consider two cases separately: where $[b]$ is the basic element, and where $[b]$ is not basic. Our first observation is that if $[b]$ is basic, then if C' satisfies (3), so does C'', and similarly for (4). Then, we see that this map j will neither contain the partial Vershiebung at τ or the partial Frobenius at τ, and hence the component C''_j will also satisfy (3) (resp. (4)).

If $[b]$ is not the basic element, then we know that j is either $\mu(p)$ or $\omega(\mu)(p)$. In this case, we see that we can commute C'', i.e. we can write C''_j as either $C'' \cdot X_J$ or $X_{J'} \cdot C''$. To see this, we consider $\pi_1(C'')$. This is an irreducible component that is a (connected component of a) generalized Goren-Oort strata. Hence, there is some set J, such that $\pi_1(C'')$ is the generalized Goren-Oort strata Y_J. A similar analysis allows us to determine J' such that $\pi_2(C'')$ is the generalized Goren-Oort strata Y_J. In particular, we claim that either

- the universal isogeny over X_J contains the partial Frobenius at $\tau \in I_{\theta,+}$ and the universal isogeny over C' does not contain the partial Vershiebung at $\tau \in I_{\theta,+}$; or
- the universal isogeny over $X_{J'}$ contains the partial Vershiebung at $\tau \in I_{\theta,-}$ and the universal isogeny over C' does not contain the partial Frobenius at $\tau \in I_{\theta,-}$

This follows from analyzing the possible $\pi_1(C')$, $\pi_2(C')$. Indeed, we see that the first case occurs when $\pi_1(C')$ lies in $\mathcal{S}_\tau$ for $\tau \in I_{h,+}$. Indeed, in this case we have $I_n = \sigma(J)^c$. We claim now that for $\tau \in I_{\theta,+}$; we must have $\tau \notin J^+ \cup J^-$. Indeed, we see that the set J consists of some groups of continuous '(', and we take the left-most term of each set of continuous '('. This set must be contained in $I_{h,-} \cup I_{\theta,-}$. We now want to show that there exists some $\tau \in I_{\theta,+}$, such that at τ we have a ')' instead. We take the τ with the largest indexed number, and we see that this will satisfy the desired condition. We also see from the above analysis that C' cannot contain the partial Vershiebung at τ.

Similarly, we see that the second case occurs when $\pi_2(C')$ lies in $\mathcal{S}_\tau$ for $\tau \in I_{h,-}$.

We now consider the case (b). For this, we claim that we will always get a component C such that will satisfy either (1) or (2). For $n = 2$, this follows from seeing that one can write the composition $X_{I_1} \cdot X_{I_2}$, and observe that we will have that if $\tau \in I_2^+$, then all of $\tau, \sigma^{-1}\tau, \sigma^{-2}\tau$ does not in I_1 or $\sigma\tau, \tau, \sigma^{-1}\tau$ lies in I_1. For $n > 2$, we will have $C' = X_{I_1,j}$ for some I_1, j, and a similar argument holds, since X_{I_1} and $X_{I_1,j}$ are related by a Frobenius factor. □

In fact, we can say more about the map $T_{p^n,I}$:

Proposition 7.2.3. *The support of $T_{p^n,I}$ is $X_{I,n}$.*

Proof. We can look at the restriction to the central leaf, and we see that the map is given by an element of $j \in J_b(\mathbb{Q}_p)$ which must be either $\omega(\mu)^n(p)$ or $(\mu)^n(p)$. However, we see that if it is $\omega(\mu)^n(p)$, then it will be zero after applying $\theta \in I_{\theta,+}$ as in the proof of the previous proposition. □

Proposition 7.2.4. *The maps $T_{p^n,I}$ and the composition*

$$R\Gamma(\mathcal{S}, \omega^{1-I_{\theta,-}-I_{h,-}}) \to R\Gamma(\mathcal{S}, \omega^{1-I_{\theta,-}-I_{h,-}+I_{\theta,+}+I_{h,+}}) \xrightarrow{U_{\mathfrak{p}^n}} R\Gamma(\mathcal{S}, \omega^{1-I_{\theta,-}-I_{h,-}+I_{\theta,+}+I_{h,+}}) \to R\Gamma(\mathcal{S}, \omega^{1+I_{\theta,+}+I_{h,+}})$$

agree.

Proof. To show equality, we need to compare the maps over the open $X^{\circ}_{I,n}$. But this follows from the fact that we have a local isomorphism $p_1^*\omega^\kappa \xrightarrow{\sim} p_2^*\omega^\kappa$ and that pulling back of the differential operator ∇_τ is still a differential operator, which is compatible with the above isomorphism. □

7.3. **Extending the Jacquet-Langlands map.**

7.3.1. We now want to construct a map which mimics the transfer map we have in §5.2 except that we start in an irregular weight (weight one). As in that situation, we want to construct a cohomological correspondence defined over X_I. Note that we have one defined over the open KR-strata by restricting the T_p-correspondence, and we would like to extend over the closure, to get an actual cohomological correspondence supported on X_I. This may not be possible, so we instead try to construct the diagonal map in the diagram:

$$\begin{array}{ccccc} R\Gamma(\mathcal{S},\omega^\kappa) & \xrightarrow{\iota_2^*} & R\Gamma(\mathcal{S}_{n,I,-},\iota_2^*\omega^\kappa) & \xleftarrow{\iota_{1,2!}} & R\Gamma(Z_I,\omega^\kappa|_{Z_I}\otimes \det\mathcal{N}_{I,+}[c]) \\ & & & \overset{T_{p,I}}{\searrow} & \downarrow T_{Z_I^n} \\ & & & & R\Gamma(Z_I,\omega^\kappa|_{Z_I}\otimes \det\mathcal{N}_{I,+}[c]). \end{array}$$

Thus, we want a correspondence defined over the closed stratum given by $p_1^{-1}(Z_I)\subset X_I$, and we know that we have a correspondence over $p^{-1}(Z_I)\cap X_I^\circ$. Our first observation is that after applying various partial Hasse invariants or partial theta operators, the map extends, and we want to show that this implies that the map extends. Moreover, we know that these partial Hasse invariants/partial theta operators are all isomorphisms on Z_I°. We want to say that the locus $(p_2^{-1}(V(h_\tau)\cap Y_{I^+})\cup X_I^\circ)\cap p^{-1}(Z_I^\circ)$ for $\tau\in I_{\theta,-}$ is of codimension at least 2 in $p^{-1}(Z_I^\circ)$.

Assuming this is true, we see that we can extend the map over $p_1^{-1}(Z_I^\circ)$. This is because we observe that we can extend the map after pre-composing with θ_τ, and we know that over the non-vanishing locus of h_τ, the map is given by the differential operator ∇_τ. In particular, we see that after composing with a differential operator has no poles, hence this must also be true before composing. Thus, we see that we can extend the map over the locus $p_2^{-1}(Y_I\backslash V(h_\tau))$, so it remains to observe that since $p_1^{-1}(Z_I^\circ)$ is Cohen-Macaulay, we will be able to extend this map of line bundles over $p_1^{-1}(Z_I^\circ)$.

We now want to show the claim about codimensions. To see this, note that $X_I\backslash X_I^\circ$ consists of components of the form $X_I\cap X_{I'}$, where I' and I differ by a single element. It suffices to show that the each of these $X_I\cap X_{I'}$ is not entirely contained in $V(h_\tau)$. Observe that we always have $\tau\notin I$, and $\sigma^{-1}\tau\in I$, hence if we want to consider a set I' with $p_2(X_{I'})\subset V(h_\tau)$, then the set I,I' would differ at at least two places.

We denote this induced cohomological correspondence by $T_{p,I}$.

7.4. **Composition with $T_\mathfrak{p}$.**

We can now further consider the cohomological correspondence obtained via composing with $T_\mathfrak{p}$, as the composition

$$\begin{array}{ccccccc} R\Gamma(\mathcal{S},\omega^\kappa) & \xrightarrow{T_\mathfrak{p}} & R\Gamma(\mathcal{S},\omega^\kappa) & \xrightarrow{\iota_2^*} & R\Gamma(\mathcal{S}_{n,I,-},\iota_2^*\omega^\kappa) & \xleftarrow{\iota_{1,2!}} & R\Gamma(Z_I,\omega^\kappa|_{Z_I}\otimes \det\mathcal{N}_{I,+}[c]) \\ & & & & & \overset{T_{p,I}}{\searrow} & \downarrow T_{Z_I^n} \\ & & & & & & R\Gamma(Z_I,\omega^\kappa|_{Z_I}\otimes \det\mathcal{N}_{I,+}[c]). \end{array}$$

We claim that the induced map from $R\Gamma(\mathcal{S}, \omega^1)$ to $R\Gamma(Z_I, \omega^\kappa|_{Z_I} \otimes \det \mathcal{N}_{I,+}[c])$ is exactly given by $T_p \circ X_I^n$. To see this, we first recall from Theorem 7.1.2 the description of the composition $T_p \cdot X_I^n$.

Proposition 7.4.1. *The $[b]$-dense components of the product $T_p \cdot X_I^n$ is of the form $X^n_{I,\mu(p)}$ and $X^n_{I,\omega(\mu)(p)}$.*

We also want to be able to define cohomological correspondences supported exactly on $T_p \cdot X_I^n$. Our first observation is that the components of $T_p \cdot X_I^n$ are contained in $T_{p^{n+1}}$. Moreover, from Proposition 7.2.2, we see that the support of the composition $T_{p^n,I}$ must also contain the $[b]$-dense components of $T_p \cdot X_I^n$. We now claim that; at least restricting to the open $[b]$-Newton strata, all these components are actually disjoint.

To see this, we first recall from the description of irreducible components Proposition 7.1.1 that we may write all components as $\tilde{\pi}_\infty(X, Y, jZ)$, for some fixed X, Z and letting Y vary over various connected components of the Igusa variety, and $j \in J_b(\mathbb{Q}_p)$. It remains to see that jZ, for j corresponding to different $J_b(\mathbb{Q}_)/J_b(\mathbb{Z}_p)$ are disjoint.

In particular, by restricting the correspondence T_{p^n}, and arguing as in the previous subsection, we can construct a commutative diagram

$$\begin{array}{ccccc} R\Gamma(\mathcal{S}, \omega^\kappa) & \xrightarrow{\iota_2^*} & R\Gamma(\mathcal{S}_{n,I,-}, \iota_2^*\omega^\kappa) & \xleftarrow{\iota_{1,2!}} & R\Gamma(Z_I, \omega^\kappa|_{Z_I} \otimes \det \mathcal{N}_{I,+}[c]) \\ & & & \overset{(T_p \cdot X_I^n)}{\searrow} & \downarrow ? \\ & & & & R\Gamma(Z_I, \omega^\kappa|_{Z_I} \otimes \det \mathcal{N}_{I,+}[c]). \end{array} \tag{7.4.2}$$

where the dashed arrow is constructed from $T_p \cdot X_I^n$. We now claim that the two maps we have constructed from $R\Gamma(\mathcal{S}, \omega^1)$ to $R\Gamma(Z_I, \omega^\kappa|_{Z_I} \otimes \det \mathcal{N}_{I,+}[c])$ agree.

Proposition 7.4.3. *The two maps*

$$R\Gamma(\mathcal{S}, \omega^\kappa) \xrightarrow{\iota_2^*} R\Gamma(\mathcal{S}_{n,I,-}, \iota_2^*\omega^\kappa) \xrightarrow{(T_p \cdot X_I^n)} R\Gamma(Z_I, \omega^\kappa|_{Z_I} \otimes \det \mathcal{N}_{I,+}[c])$$
$$R\Gamma(\mathcal{S}, \omega^\kappa) \xrightarrow{T_p} R\Gamma(\mathcal{S}, \omega^\kappa) \xrightarrow{\iota_2^*} R\Gamma(\mathcal{S}_{n,I,-}, \iota_2^*\omega^\kappa) \xrightarrow{(X_I^n)} R\Gamma(Z_I, \omega^\kappa|_{Z_I} \otimes \det \mathcal{N}_{I,+}[c])$$

Proof. We only need to check equality on an open, from which this readily follows since we know that we have equality of supports, and the correspondences are determined by their supports. □

We can also further determine what the map labelled ? in the diagram is. We first observe that the underlying algebraic correspondence is in fact just the union of the two μ-ordinary components

$$U^{I,n}_{p,\mu(p)} \qquad \text{and} \qquad U^{I,n}_{p,\omega\mu(p)}.$$

In fact, we can now see the following:

Proposition 7.4.4. *The cohomological correspondence inducing the map labelled '?' in* (7.4.2) *is in fact the sum of two correspondences:*

$$U_p^{I,2n} \qquad \textit{and} \qquad \langle p^{2n} \rangle.$$

Proof. Again, it suffices to check over the open. For this, we may argue as in §5.2.11. Here, we note that over the open of $X^n_{I,\mu(p)}$, we again see that the map is locally the composition of an isomorphism with the inverse Cartier isomorphism (for the p^n-Frobenius). For $X^n_{I,\omega\mu(p)}$, we see that for τ in I, the map is simply an isomorphism, since locally we see that for p_1 locally the map on differentials is an isomorphism, and so is the map $p_2^*\omega \to p_1^*\omega$, since we see that locally the map is also an isomorphism, since it will be given by $V_\tau \circ V_\tau^{-1}$. □

8. Proof of Main Theorem

8.1. **Duality.** Our first step is to take duals so that we may always assume that $\#I < \#I^c$. For this, we recall that by Serre duality, we have an isomorphism

$$H^i(\overline{\mathrm{Sh}}^{\mathrm{tor}}, \omega^\kappa) \simeq H^{d-i}(\overline{\mathrm{Sh}}^{\mathrm{tor}}, \omega^{\kappa^\vee}(-D))^\vee,$$

where D is the toroidal boundary, and $\kappa^\vee := ((2-\kappa_\tau), k)$ is the dual weight of κ.

To see this, we will recall some key results about how $T_\mathfrak{p}$ interacts with duality. We first recall the following result [ERX17b, Lemma 2.10], about how Serre duality intertwines with the action of the Hecke operators.

Proposition 8.1.1. *We have an isomorphism of cohomological correspondences*

$$T_\mathfrak{p}^\vee = S_\mathfrak{p}^{-1} \circ T_\mathfrak{p}.$$

In particular, we see that this implies that if we start with some v which is an eigenvector for all T_ℓ, $\ell \neq p$, as well as $T_\mathfrak{p}$ and $S_\mathfrak{p}$, then the dual will also satisfy these conditions.

8.2. **Lifting eigenclasses.** We now want to combine the analysis in §6 and §7 to show that there is some I so that we can lift to a section of some line bundle on Z_I. Recall the key commutative diagram

$$\begin{array}{ccccc} R\Gamma(\mathcal{S}, \omega^\kappa) & \xrightarrow{\iota_2^*} & R\Gamma(\mathcal{S}_{n,I,-}, \iota_2^*\omega^\kappa) & \xleftarrow{\iota_{1,2!}} & R\Gamma(Z_I, \omega^\kappa|_{Z_I} \otimes \det \mathcal{N}_{I,+}[c]) \\ & & & \overset{T_{p,I}}{\searrow} & \downarrow T_{Z_I^n} \\ & & & & R\Gamma(Z_I, \omega^\kappa|_{Z_I} \otimes \det \mathcal{N}_{I,+}[c]). \end{array} \tag{8.2.1}$$

Proposition 8.2.2. *Let v be an eigenclass in $H^j(\mathcal{S}, \omega^1)$. Let I be the set as in Corollary 6.2.5. We assume that $\#I < \#I^c$, otherwise we may take the dual. Then, there is some class $w \in H^{j-c}(Z_I, \omega^\kappa|_{Z_I} \otimes \det \mathcal{N}_{I,+})$ lifting v along the top row in* (8.2.1).

Proof. We first claim there exists some positive integer n such that $n \geq N_I$ and $T_{p^n}(v) \neq 0$. Note that since we assumed that v was an eigenvector for both T_p and S_p, this implies that $T_{p^n}(v) = av$ for some non-zero constant v. Such an n will exist because suppose that we have for some $n \geq N_I$, $T_{p^n}(v) = 0$. Observe that we have $T_{p^{2n}} = T_{p^n} \circ T_{p^n} - 2p^n \cdot 1_{pK_p}$, and since we know that $S_p(v) \neq 0$, since it is an isomorphism, this implies that $T_{p^{2n}}(v) \neq 0$.

Applying Corollary 6.2.5, we see that there exists some class $v' \in H^j(\mathcal{S}, \omega^{1-I_{\theta,-}-I_{h,-}})$ such that v is the image of v' after applying the partial θ and Hasse operators, and there is some class $v'' \in H^j(\mathcal{S}, \omega^{1+I_{\theta,-}+I_{h,-}})$ which is the image of v under the partial Hasse and theta operators.

Now, observe that we thus have $T_{p^n,I}(v') = av''$, for some non-zero constant a. From Proposition 7.2.2, we see that the cohomological correspondence can be seen as supported on some algebraic correspondence C such that $p_1(C) = Y_{I,+}$ and $p_2(C) = Y_{I,-}$. As a result, we can factor $T_{p^n,I}$ as the composition

$$R\Gamma(\mathcal{S}, \omega^{1-I_{\theta,-}-I_{h,-}}) \to R\Gamma(Y_{I,-}, p_2^*(\omega^{1-I_{\theta,-}-I_{h,-}})) \xrightarrow{T_{\mathfrak{p}^n}} R\Gamma(Y_{I,+}, p_1^!(\omega^{1+I_{\theta,+}+I_{h,+}}) \to R\Gamma(\mathcal{S}, \omega^{1+I_{\theta,+}+I_{h,+}}),$$

and in particular we see that the restriction of v' to $Y_{I,-}$ is non-zero, and similarly v'' lies in the image of the pushforward map from $Y_{I,+}$. Finally, we see that we have a commutative diagram

$$\begin{array}{ccccc} H^j(\mathcal{S},\omega^1) & \longrightarrow & H^j(Y_{I-},\omega^1) & \longleftarrow & H^0(Z_I,\omega_I) \\ \uparrow & & \uparrow & & \uparrow \\ H^j(\mathcal{S},\omega^{1-I_{\theta,-}-I_{h,-}}) & \longrightarrow & H^j(Y_{I-},\omega^{1-I_{\theta,-}-I_{h,-}}) & \longleftarrow & H^0(Z_I,\omega_I) \\ \downarrow & & \downarrow & & \downarrow \\ H^j(\mathcal{S},\omega^{1-I_{\theta,-}-I_{h,-}+I_{\theta,+}+I_{h,+}}) & \longrightarrow & H^j(Y_{I-},\omega^{1-I_{\theta,-}-I_{h,-}+I_{\theta,+}+I_{h,+}}) & \longleftarrow & H^0(Z_I,\omega_I), \end{array}$$

where on the rightmost column all the maps are isomorphisms, and we observe that we can lift the image of v' along the bottom row. □

Proposition 8.2.3. *Let w be as in Proposition 8.2.2, and a the non-zero constant such that $T_{p^{2n}}(v) = av$. Then, we have the equality*

$$U_p^{I,2n}(w) - aU_p(w) - S_p^n(w) = 0.$$

Proof. This follows from the commutativity of (7.4.2), and Proposition 7.4.4, as applied to the class v. □

As a corollary, we can now see that w, and hence v, must be attached to an ordinary Galois representation.

Corollary 8.2.4. *We have that $w \in e(U_p)H^{j-c}(Z_I, \omega^\kappa|_{Z_I} \otimes \det \mathcal{N}_{I,+})$.*

8.3. **Doubling.** We now want to construct a two dimensional subspace of ordinary classes. The way to do this is by applying U_p^I to w; we claim that this image is linearly independent of w.

8.3.1. We now need to show that the image of $w, U_p^{I,n}(w)$ in $H^{j-c}(Z_I, \omega^\kappa|_{Z_I} \otimes \det \mathcal{N}_{I,+})$ are linearly independent.

For simplicity, we consider the case where p is inert; the general case is similar. Suppose on the contrary that we have $U_p^{I,n}(w)$ being a constant multiple of w. We want to show that this also implies that w is an eigenvector of $F_p^{I,n}$, the dual to U_p^I. Indeed, we observe that this is true since w is an eigenvector for $U_p^I + F_p^I$. We will now show that the extension of w on $D = Z_I \backslash Z_I^\circ$ is a zero of order at least p. To see this, observe that by assumption w has to lie in the image of

$$H^0(Z_I, \kappa_I') \xrightarrow{\mathrm{Frob}_q} H^0(Z_I, \kappa_I),$$

where κ_I' is of weight one at $\tau \notin (I^+ \cup I^-)$. Thus, we see that since w is an eigenvector, we must have that w has a zero on $Z_I \backslash Z_I^\circ$, and since it is the image of Frobenius, the order of the zero must be a multiple of p. Thus, we see that in fact we have that w lies in the image of $H^0(Z_I, \kappa_I'(-D))$.

Iterating this argument, and observing that F_p^I satisfies that $p_1(D) = qp_2(D)$, we hence see that w always lies in the image of the map

$$H^0(Z_I, \omega^{\kappa - N(p-1)}) \xrightarrow{\cdot h_{I,\mathfrak{p}}^N} H^0(Z_I, \omega^\kappa),$$

for all integers $N > 0$. However, we see that this is impossible, since this implies that the cohomology group $H^0(X, \omega^{\kappa+N(p-1)}(-D))$ is nonzero for all $N > 0$, which is clearly impossible since $H^0(Z_I, \omega(-ND))$ is eventually zero.

Now that we have established $\mathfrak{p}$-ordinarity, and doubling, we can proceed to show that that Galois representation is unramified.

8.3.2. We now adapt the argument of [DW18], to show that since w, U_p^I lies in the ordinary part of $H^0(Z_I, \omega^{\underline{\kappa}})$, then the Galois representation ρ_c attached to c is unramified.

We first observe that the image of $v \in H^j(\mathcal{S}, \omega^{(\mathbf{1},-\mathbf{1})})$ does not lift to a constant mod p modular forms of Z_I. Indeed, observe that the constant functions in $H^0(Z_I, \omega^{\underline{\kappa}_I})$ map to zero in $H^j(Y_{I-}, \omega^1)$ since they are the restriction of the constant functions.

Now, we denote by $W := \mathbb{F}_p \cdot w \oplus \mathbb{F}_p \cdot U_p^I(w)\}$ the U_p^I span of w. Note that it must be two dimensional, since we have established that $U_p^I(w), w$ are linearly independent, and moreover the polynomial relation from Proposition 8.2.3 shows that this space is exactly two dimensional.

Assuming this, we may conclude as follows. Consider the minimal polynomial $\tilde{H}$ of U_p^I acting on W. Thus, we have two possibilities: All the roots of $\tilde{H}$ are equal, or we have at least two distinct roots.

At least 2 distinct roots: The argument for this is standard, and follows exactly as in [ERX17a], [DW18]. Indeed, let α,β be two distinct roots of $\tilde{H}$. Then, we may choose some element $f_\alpha \in W$, with the same prime to S Hecke eigenvalues, which moreover has T_p-eigenvalue given by α. Then there exists a lift of f_α to a $\mathbb{T}_S[T_p]$-eigenform $\tilde{f}_\alpha$ whose eigenvalues lift those of f_α, in particular $\tilde{f}_\alpha$ is p-ordinary with $T_p(\tilde{f}_\alpha) = \tilde{\alpha}_p f_\alpha$ for some p-adic unit $\tilde{\alpha}$. Moreover, since the operators S_q for $q \notin S$ commute with the action of T_l we can assume in addition that $\tilde{f}_\alpha$ has central character $\tilde{\epsilon}$ lifting ϵ.

We have the following result of Wiles:

Proposition 8.3.3. *$V(\tilde{f}_\alpha)$ has a 1 dimensional quotient such that the decomposition group D_p of p acts on it by the unramified character sending* Frob_p *to the unique unit root of $X^2 - \alpha_p X + \tilde{\epsilon}(p)\mathrm{N}_{F/\mathbb{Q}}(p)^{k-1}$.*

In particular, taking reduction mod p, we have the following proposition:

Proposition 8.3.4. *For any non-constant f_α as above, the representation $\rho_{f_\alpha}|_{D_p}$ admits a 1-dimensional unramified quotient on which* Frob_p *acts by α.*

Since $\alpha \neq \beta$, this shows that we have two non-equal 1-dimensional unramified subquotients of $\rho_c|_{D_p}$, which implies ρ_c is unramified. We also remark that this shows that $\tilde{H}$ can have at most 2 distinct roots.

We now consider the case where $\tilde{H}$ has exactly one root. This is essentially the argument in [DW18], we will adapt it to this situation.

Lemma 8.3.5. *T_p^κ does not act semisimply on W.*

Proof. Observe that if T_p^κ acted semisimply on W, then the minimal polynomial of T_p^κ would be constant. However, we know the minimal polynomial of T_p^κ acting on W is of degree at least 2. □

Now, suppose that ρ_c is irreducible and $\tilde{H}(X)$ has a multiple root α. If ρ_c is not irreducible, then it is the sum of two characters $\chi_1 \oplus \chi_2$. Moreover, we observe that the determinant of ρ_c is unramified at p, and given by $\chi_1\chi_2$. Since we know that ρ_c has an unramified quotient, which must correspond to one of these χ_i's, we see that both χ_1 and χ_2 are unramified at p, and hence ρ_c is unramified at p.

Denote by $\mathfrak{m}'$ the maximal ideal of $\mathbb{T}'$ corresponding to c . Moreover, let $\mathfrak{m}$ be the maximal ideal of $\mathbb{T}'[T_p]$ corresponding to c and the choice of α.

Let $\mathbb{T}'_\kappa$ be the image of $\mathbb{T}'$ in the ring of endomorphisms of $H^0(\mathrm{Sh}^{\mathrm{tor}}_{\mathbb{Z}_p}, \omega^{\underline{\kappa}})_\mathfrak{m}$. In particular, the algebra generated by $\mathbb{T}'$ acting on $H^0(\mathrm{Sh}^{\mathrm{tor}}_{\mathbb{F}_p}, \omega^{\underline{\kappa}})_\mathfrak{m}$ is naturally isomorphic to $\mathbb{T}'_\kappa \otimes_{\mathbb{Z}_p} \mathbb{F}_p$. (Large enough κ such that $H^0(\mathrm{Sh}^{\mathrm{tor}}_{\mathbb{Z}_p}, \omega^{\underline{\kappa}})$ is torsion-free).

Moreover, we can write $\mathbb{T}' \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ as

$$\mathbb{T}' \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \simeq \prod_{g \in \mathcal{N}} \mathbb{Q}_p,$$

where $\mathcal{N}$ is the set of newforms of weight κ, and $\mathbb{T}'$ is a free $\mathbb{Z}_p$-lattice in this.

Lemma 8.3.6. *The Hecke operator T_p^κ acting on $M_\kappa(n; \mathbb{F}_p)_\mathfrak{m}$ does not belong to $\mathbb{T}' \otimes_{\mathbb{Z}_p} \mathbb{F}_p$, and hence does not belong to $\mathbb{T}'$. However, T_p^κ belongs to $\mathbb{T}' \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$.*

Proof. The proof is exactly as in [DW18, Lem 4.8], we reproduce it here for completeness. Strong Multiplicity One applied to each $g \in \mathcal{N}$ implies that $T_p^\kappa \in \mathbb{T}' \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$.

We have a $\mathbb{T}'[T_p]$-equivariant isomorphism

$$M_k(n; \mathbb{Z}_p)_\mathfrak{m} \otimes \mathbb{F}_p \simeq M_k(n; \mathbb{F}_p)_\mathfrak{m},$$

thus if T_p^κ belonged to $\mathbb{T}'$ acting on $M_k(n; \mathbb{Z}_p)\mathfrak{m}$, then T_p^κ acting on $M_k(n; \mathbb{F}_p)_\mathfrak{m}$ would belong to $\mathbb{T}'_\kappa \otimes \mathbb{F}_p$ and we will now show that this is impossible.

Define W as previously. Then W is a $\mathbb{T}'[T_p]$-stable subspace of $M_k(n; \mathbb{F}_p)\mathfrak{m}$. Hence, if T_p^κ belonged to $\mathbb{T}'_k \otimes \mathbb{F}_p$, T_p^k would also belong to $\mathbb{T}'$ acting on W. However $\mathbb{T}'$ acts on W by a character, whereas by the previous lemma the action of T_p^κ on W is not semisimple. □

We see from the construction of ρ_c by [Car94, Théorème 2], that there exists a free of rank two $\mathbb{T}'_\mathfrak{m}$- module $\mathcal{M}$ with a continuous action of Gal_F such that the Gal_F action on $\mathcal{M}$ induces a GF -equivariant isomorphism

$$\mathcal{M} \otimes_\mathcal{O} K \simeq \prod_{g \in \mathcal{N}} V(g)$$

where $V(g)$ denotes the $K[D_p]$-module corresponding to the Galois representation attached to g. Note that by construction, by the previous result of Wiles, one has a short exact sequence of $K[D_p]$-modules

$$0 \to V(g)^- \to V(g) \to V(g)^+ \to 0.$$

where $V(g)^+$ and $V(g)^-$ have dimension 1 over K. Moreover, D_p acts on $V(g)^-$ via an unramified character mapping Frob_p to $\tilde{\alpha}$, a lift of α.

Now, we define

$$\mathcal{M}^+ := \mathcal{M} \cap \prod_{g \in \mathcal{N}} V(g)^+$$

and let

$$\mathcal{M}^- := \mathrm{Im}(\mathcal{M} \to \prod_{g \in \mathcal{N}} V(g)^-).$$

This gives us a short exact sequence of $\mathbb{T}'_\mathfrak{m}$-modules

$$0 \to \mathcal{M}^+ \to \mathcal{M} \to \mathcal{M}^- \to 0,$$

and we want to show that the unramified mod $\mathfrak{m}$ quotient $\mathcal{M}^-/\mathfrak{m}\mathcal{M}^-$ is 2 dimensional, hence equal to $\mathcal{M}/\mathfrak{m}\mathcal{M}$.

Since $\mathcal{M}^-$ is not 0, by Nakayama's lemma for the local O-algebra $\mathbb{T}'_{\mathfrak{m}}$, the $\mathbb{F}[D_p]$-module $\mathcal{M}^-/\mathfrak{m}\mathcal{M}^-$ is not 0. Thus, it remains to show $\mathcal{M}^-/\mathfrak{m}\mathcal{M}^-$ cannot be one dimensional. We will show this by contradiction.

Suppose that $\mathcal{M}^-/\mathfrak{m}\mathcal{M}^-$ is one dimensional. Then, since $\mathcal{M}^-$ is a free $\mathcal{O}$-module of the same rank as $\mathbb{T}'_{\mathfrak{m}}$, Nakayama's lemma implies that $\mathcal{M}^-$is a quotient of $\mathbb{T}'_{\mathfrak{m}}$, and hence isomorphic to $\mathbb{T}'_{\mathfrak{m}}$. Moreover, we know that the uniformizer ϖ_p acts on $\mathcal{M}^-$ via an element $U \in \mathbb{T}'_{\mathfrak{m}}$. Since taking modulo $\mathfrak{m}$, U maps to the Hecke operator T_p^κ, which implies that $T_p^\kappa \in \mathbb{T}'_{\mathfrak{m}}$, a contradiction to the Lemma above.

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